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KINEMATICAL LIE ALGEBRAS VIA DEFORMATION THEORY

JOSÉ M. FIGUEROA-O'FARRILL

ABSTRACT. We present a deformation theory approach to the classification of kinematical Lie algebras in $3 + 1$ dimensions and present calculations leading to the classifications of all deformations of the static kinematical Lie algebra and of its universal central extension, up to isomorphism. In addition we determine which of these Lie algebras admit an invariant symmetric inner product. Among the new results, we find some deformations of the centrally extended static kinematical Lie algebra which are extensions (but not central) of deformations of the static kinematical Lie algebra. This paper lays the groundwork for two companion papers which present similar classifications in dimension $D + 1$ for all $D \geq 4$ and in dimension $2 + 1$.

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1. INTRODUCTION

The study of kinematical Lie algebras is intimately linked to the principle of relativity, which may be interpreted as a physical avatar of Klein's Erlanger Programme, by which a geometry can be studied via its Lie group of automorphisms. In the physical context, we may say that geometrical models of the universe are determined by their relativity group. As in Klein's programme, by geometry one does not necessarily mean a metric geometry, but any geometrical data which the automorphisms leave invariant. For example, the Newtonian model of the universe is an affine bundle (with three-dimensional fibres to be interpreted as *space*) over an affine line (to be interpreted as *time*) and it has the galilean group as automorphisms, whose invariant notions are the time interval between events and the euclidean distance between simultaneous events. By contrast, Minkowski spacetime has the Poincaré group as the group of automorphisms and the invariant notion is the proper distance (or, equivalently, the proper time), which defines a lorentzian metric. Both the galilean and Poincaré groups are examples of kinematical Lie groups, whose Lie algebras (in dimension $3 + 1$) are the subject of this paper.

Parenthetically, it is an unfortunate misnomer in Physics that we use the word "relativistic" only in the case of Poincaré relativity and "non-relativistic" in other cases: they are all in a strict sense relativistic, only that the relativity group is different.

By a **kinematical Lie algebra** in dimension D , we mean a real $\frac{1}{2}(D+1)(D+2)$ -dimensional Lie algebra with generators $R_{ab} = -R_{ba}$, with $1 \leq a, b \leq D$, spanning a Lie subalgebra isomorphic to $\mathfrak{so}(D)$:

$$[R_{ab}, R_{cd}] = \delta_{bc}R_{ad} - \delta_{ac}R_{bd} - \delta_{bd}R_{ac} + \delta_{ad}R_{bc}, \quad (1)$$

and B_a , P_a and H which transform according to the vector, vector and scalar representations of $\mathfrak{so}(D)$, respectively – namely,

$$\begin{aligned} [R_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\ [R_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [R_{ab}, H] &= 0. \end{aligned} \quad (2)$$

The rest of the brackets between B_a , P_a and H are only subject to the Jacobi identity: in particular, they must be $\mathfrak{so}(D)$ -equivariant. The kinematical Lie algebra where those additional Lie brackets vanish is called the **static** kinematical Lie algebra, of which, by definition, every other kinematical Lie algebra is a deformation.

Up to isomorphism, there is only one kinematical Lie algebra in $D = 0$: it is one-dimensional and hence abelian. For $D = 1$, there are no rotations and hence any three-dimensional Lie algebra is kinematical. The classification is therefore the same as the celebrated Bianchi classification of three-dimensional real Lie algebras [1]. The classification for $D = 3$ is due to Bacry and Nuyts [2] who completed earlier work of Bacry and Lévy-Leblond [3]. The present paper presents a deformation theory approach to this classification, based on earlier work [4] for the galilean and Bargmann algebras, and also the classification of deformations of the universal central extension of the static kinematical Lie algebra. This paper is also intended to lay the groundwork to two further papers: in [5] we classify the kinematical Lie algebras for $D > 3$ with and without central extension, and in [6] we classify kinematical Lie algebras for $D = 2$. Despite sharing the same methodology, the problems differ sufficiently in the technicalities to merit them being split. A summary of the results in this series of papers can be found in [7].

An important characteristic of Lie algebras, particularly for physical applications, is whether or not they admit an invariant inner product, by which in this paper we mean an invariant nondegenerate symmetric bilinear form. Such Lie algebras are said to be **metric**. Cartan's semisimplicity criterion says that the Killing form of a semisimple Lie algebra is an invariant inner product. At the other extreme, any inner product on an abelian Lie algebra is invariant. For each of the kinematical Lie algebras in the paper we determine which ones are metric.

This paper is organised as follows. In Section 2 we set the notation by reviewing the basic notions of Lie algebra deformations, following in spirit the seminal work of Nijenhuis and Richardson [8]. We recall the definition of a graded Lie superalgebra structure on the space $A^\bullet = \Lambda^{\bullet+1}V^* \otimes V$ of alternating multilinear maps from a vector space V to itself and identify Lie algebra structures on V in terms of this Lie superalgebra. We discuss the Maurer–Cartan equation giving rise to deformations of a Lie algebra structure on V and discuss the perturbative solution of the Maurer–Cartan equation, which lies at the heart of the deformation-theory approach to Lie algebra classifications. We introduce the notions of infinitesimal deformations and of obstructions to integrating an infinitesimal deformation and how both can be rephrased cohomologically. Section 3 applies this technology to recover the Bacry–Nuyts classification of kinematical Lie algebras in dimension $3 + 1$. The results are summarised in Table 1. A crucial rôle is played by the automorphisms of the static Lie algebra which preserve the deformation complex,

so we pay considerable attention at how such automorphisms decompose the space of cochains. The decomposition of the space of cochains into sub-modules of the group of automorphisms is important in order to make a convenient choice for parametrising the infinitesimal deformations. The decomposition of the space of cocycles into orbits of the group of automorphisms is crucial in the solution of the obstruction equations and thus in determining the integrability locus. Finally, automorphisms play a rôle in bringing deformations to normal forms so that we can determine when two deformations are isomorphic. In Section 4 we apply this methodology to classify deformations of the universal central extension of the static kinematical Lie algebra. The results here are summarised in Table 2: they include some well-known Lie algebras (central extensions of kinematical Lie algebras) and some less well-known Lie algebras which are non-central extensions of kinematical Lie algebras. Section 5 offers some conclusions. The paper contains three appendices. In Appendix A we review the basic notions of Chevalley–Eilenberg cohomology, whereas in Appendices B and C we provide details of our choice of bases for the deformation complexes, which should allow any interested party in reproducing our results.

2. DEFORMATION THEORY OF LIE ALGEBRAS

In this section we review the basic notions of the deformation theory of Lie algebras, first introduced by Nijenhuis and Richardson in [8].

2.1. The graded Lie superalgebra of alternating maps. We start by recalling the definition of a graded Lie superalgebra structure on the space of alternating multilinear maps.

Let V be a (finite-dimensional, real) vector space and let V^* denote its dual. Let $A^p = \Lambda^{p+1}V^* \otimes V$ denote the space of skewsymmetric $(p+1)$ -multilinear maps

$$\underbrace{V \times \cdots \times V}_{p+1} \rightarrow V. \quad (3)$$

For $\alpha \in \Lambda^{p+1}V^*$, $\beta \in \Lambda^{q+1}V^*$, and $X, Y \in V$, let us define

$$(\alpha \otimes X) \bullet (\beta \otimes Y) := (\alpha \wedge \iota_X \beta) \otimes Y. \quad (4)$$

We then extend it bilinearly to define a product

$$\bullet : A^p \times A^q \rightarrow A^{p+q}. \quad (5)$$

If $\lambda \in A^p$ and $\mu \in A^q$, we define their (Nijenhuis–Richardson) bracket by

$$[\![\lambda, \mu]\!] := \lambda \bullet \mu - (-1)^{pq} \mu \bullet \lambda, \quad (6)$$

which makes it clear that it is skew-symmetric (in the super-sense):

$$[\![\lambda, \mu]\!] = -(-1)^{pq} [\![\mu, \lambda]\!]. \quad (7)$$

An easy calculation (made easier by choosing $\lambda = \alpha \otimes X$, $\mu = \beta \otimes Y$ and $\nu = \gamma \otimes Z$), shows that it also satisfies the Jacobi identity (also in the super-sense):

$$[\![\lambda, [\![\mu, \nu]\!]]\!] = [[[\lambda, \mu], \nu]] + (-1)^{pq} [\![\mu, [\![\lambda, \nu]\!]]\!], \quad (8)$$

where $\lambda \in A^p$ and $\mu \in A^q$. In other words, $(A^\bullet, [\![\cdot, \cdot]\!])$ is a graded Lie superalgebra.

Let us look more closely at the component $[\![\cdot, \cdot]\!] : A^1 \times A^1 \rightarrow A^2$. Let $\lambda, \mu \in A^1$ and let $X, Y, Z \in V$. Then a short calculation shows that

$$[\![\lambda, \mu]\!](X, Y, Z) = \lambda(\mu(X, Y), Z) + \mu(\lambda(X, Y), Z) + \text{cyclic}. \quad (9)$$

Taking $\lambda = \mu$,

$$\frac{1}{2} [\![\mu, \mu]\!] = \mu(\mu(X, Y), Z) + \text{cyclic}, \quad (10)$$

whose vanishing is the Jacobi identity for the bracket on V defined by μ . In other words, $\mu \in A^1$ defines a Lie bracket on V if and only if

$$[\![\mu, \mu]\!] = 0. \quad (11)$$

Notice that $A^0 = V^* \otimes V$ is a Lie subalgebra under $[\![\cdot, \cdot]\!]$ isomorphic to $\mathfrak{gl}(V)$ and the adjoint action of A^0 on A^\bullet , defined by $[\![\lambda, \cdot]\!]$ for $\lambda \in A^0$, is the natural action of $\mathfrak{gl}(V)$ on A^\bullet . Indeed, if $\mu \in A^p$, then

$$[\![\lambda, \mu]\!](Z_0, \dots, Z_p) = \lambda\mu(Z_0, \dots, Z_p) - \mu(\lambda Z_0, Z_1, \dots, Z_p) - \cdots - \mu(Z_0, \dots, Z_{p-1}, \lambda Z_p). \quad (12)$$

This integrates to an action of $GL(V)$, which is the group of invertible elements in A^0 , on A^\bullet as automorphisms of the Nijenhuis–Richardson bracket. Therefore if μ satisfies equation (11) so does $g \cdot \mu$, where $g \in GL(V)$ and

$$(g \cdot \mu)(X, Y) = g\mu(g^{-1}X, g^{-1}Y). \quad (13)$$

The moduli space \mathcal{M} of Lie algebra structures on V is then the space of solutions $\mu \in A^1$ to equation (11) modulo the action of $GL(V)$, for it is clear that if two Lie algebra structures are in the same orbit of $GL(V)$ then they are isomorphic. If a Lie algebra structure μ_0 lies in the *closure* of the $GL(V)$ orbit of a Lie algebra structure μ , then μ and μ_0 may or may not give rise to isomorphic Lie algebras, but in any case we say that the Lie algebra defined by μ_0 is a **contraction** of the Lie algebra defined by μ . To some extent, deformation is the inverse process to contraction.

2.2. Relationship with Chevalley–Eilenberg cohomology. Let \mathfrak{g} be a Lie algebra structure on a vector space V with Lie bracket $\mu_0 \in A^1$. Since $[\mu_0, \mu_0] = 0$, it follows from the Jacobi identity for the Nijenhuis–Richardson bracket, that the operation $[\mu_0, -] : A^p \rightarrow A^{p+1}$ squares to zero:

$$[\mu_0, [\mu_0, \lambda]] = \frac{1}{2} [[\mu_0, \mu_0], \lambda] = 0 \quad \forall \lambda \in A^p. \quad (14)$$

In fact, it is up to a sign the Chevalley–Eilenberg differential ∂ on the complex $C^\bullet(\mathfrak{g}; \mathfrak{g})$ whose definition is recalled in Appendix A. Indeed, $[\mu, -]$ on A^p agrees with $(-1)^p \partial$ on $C^{p+1}(\mathfrak{g}; \mathfrak{g})$.

Since the Chevalley–Eilenberg differential is an inner derivation of the Nijenhuis–Richardson bracket, it follows that cocycles form a subalgebra of the Lie superalgebra A^\bullet , inside which the coboundaries form an ideal. Therefore the Nijenhuis–Richardson bracket descends to the cohomology and gives $H^{\bullet+1}(\mathfrak{g}; \mathfrak{g})$ the structure of a graded Lie superalgebra (but with degree shifted by one).

2.3. Deformations of Lie algebras. Let us now consider deforming the Lie bracket μ_0 of \mathfrak{g} to $\mu = \mu_0 + \varphi$ for some $\varphi \in A^1$. Then μ will define a Lie algebra if and only if equation (11) is satisfied:

$$[\mu, \mu] = [\mu_0 + \varphi, \mu_0 + \varphi] = [\mu_0, \mu_0] + 2[\mu_0, \varphi] + [\varphi, \varphi] = 0. \quad (15)$$

Since μ_0 is a Lie bracket, we see that so is μ if and only if

$$[\mu_0, \varphi] + \frac{1}{2} [\varphi, \varphi] = 0, \quad (16)$$

and since the LHS is precisely $-\partial\varphi$, this is equivalent to φ satisfying the Maurer–Cartan equation

$$\partial\varphi = \frac{1}{2} [\varphi, \varphi]. \quad (17)$$

We can try to solve this equation perturbatively by introducing a parameter t and writing¹ $\varphi = \sum_{n \geq 1} t^n \varphi_n$, where $\varphi_n \in A^1$ for $n = 1, 2, \dots$ and imposing the Maurer–Cartan equation order by order in t , arriving at the sequence of equations:

$$\partial\varphi_n = \frac{1}{2} \sum_{m=1}^{n-1} [\varphi_m, \varphi_{n-m}], \quad (18)$$

for $n = 1, 2, \dots$

2.3.1. Infinitesimal deformations. The $n = 1$ equation is simply $\partial\varphi_1 = 0$, so that φ_1 is a cocycle. Conversely, every cocycle in $C^2(\mathfrak{g}; \mathfrak{g})$ defines a **first-order** (or **infinitesimal**) deformation. A trivial kind of infinitesimal deformation is one which is tangent to the $GL(V)$ orbit of μ_0 . Let $\beta \in A^0$ and consider $T = 1 + t\beta$, which is invertible for small t or as a formal power series in t :

$$T^{-1} = 1 - t\beta + O(t^2). \quad (19)$$

Then a short calculation reveals that

$$(T \cdot \mu_0)(X, Y) = T\mu_0(T^{-1}X, T^{-1}Y) = (\mu_0 - t\partial\beta)(X, Y) + O(t^2), \quad (20)$$

which is an example of a deformation where $\varphi_1 = -\partial\beta$ is a coboundary. Conversely, every infinitesimal deformation which is a coboundary is tangent to the $GL(V)$ orbit. Therefore the tangent space $T_{\mu_0}\mathcal{M}$ at μ_0 to the moduli space of Lie algebras is the quotient of infinitesimal deformations (i.e., cocycles) by the trivial infinitesimal deformations (i.e., coboundaries) and hence isomorphic to the cohomology $H^2(\mathfrak{g}; \mathfrak{g})$.

¹Notice that we do not impose any convergence properties on the series. We are dealing therefore with formal deformations. It will turn out, however, that the deformations found in this paper are all polynomial and, therefore, trivially convergent.

2.3.2. Obstructions to integrability. Given an infinitesimal deformation φ_1 , finding the $\varphi_{n>1}$ to arrive at a deformation is known as **integrating** φ_1 . If $\varphi_1 = -\partial\beta$, then we may just integrate it by acting with the one-parameter subgroup of $GL(V)$ generated by β , so integrating an infinitesimal deformation is only ever in question when the cohomology class of φ_1 is nonzero.

The $n = 2$ equation in (18) says that $[\varphi_1, \varphi_1]$, which is a cocycle because φ_1 is, is actually a coboundary. In other words, the cohomology class of $[\varphi_1, \varphi_1]$ in $H^3(\mathfrak{g}; \mathfrak{g})$ is the **obstruction** to integrating the infinitesimal deformation φ_1 to second order in t . This obstruction class only depends on the cohomology class of φ_1 in $H^2(\mathfrak{g}; \mathfrak{g})$. Indeed, if $\varphi_1 \mapsto \varphi_1 + \partial\beta$, then

$$[\varphi_1 + \partial\beta, \varphi_1 + \partial\beta] = [\varphi_1, \varphi_1] + \partial[\beta, \varphi_1 + \partial\beta]. \quad (21)$$

This situation persists to higher order. Suppose that we have managed to integrate the deformation to order t^n , so that $\mu = \mu_0 + \sum_{k=1}^n t^k \varphi_k$ satisfies

$$[\mu, \mu] \in O(t^{n+1}). \quad (22)$$

The Jacobi identity for the Nijenhuis–Richardson bracket says that

$$[\mu, [\mu, \mu]] = 0, \quad (23)$$

so that

$$0 = [\mu_0 + \varphi, [\mu, \mu]] = \partial[\mu, \mu] + [\varphi, [\mu, \mu]]. \quad (24)$$

But $\varphi \in O(t)$ and $[\mu, \mu] \in O(t^{n+1})$, so that $\partial[\mu, \mu] \in O(t^{n+2})$ and hence the term of order t^{n+1} in $[\mu, \mu]$ is a cocycle and defines a class in $H^3(\mathfrak{g}; \mathfrak{g})$. We can integrate the deformation to order t^{n+1} precisely when the class is trivial and the cocycle is a coboundary, say, $\partial\varphi_{n+1}$.

In summary, we get a sequence of obstructions in $H^3(\mathfrak{g}; \mathfrak{g})$ to integrating the infinitesimal deformation φ_1 , and the obstructions only depend on the cohomology class of φ_1 in $H^2(\mathfrak{g}; \mathfrak{g})$.

2.3.3. Methodology. Our approach to the classification of deformations of a given Lie algebra \mathfrak{g} will therefore consist, first of all, in calculating $H^2(\mathfrak{g}; \mathfrak{g})$. This is simplified by the use of the Hochschild–Serre spectral sequence, as briefly recalled in Appendix A.2. For each class in $H^2(\mathfrak{g}; \mathfrak{g})$ we choose a cocycle representative χ_i , say, and then consider the most general linear combination $t_1\chi_1 + \dots + t_N\chi_N$ of such cocycles and determine the loci in the parameter space $\mathbb{R}^N \ni (t_1, \dots, t_N)$ corresponding to the integrable deformations. A priori this could be an infinite process, but we will see that all deformations in this paper are either quickly obstructed or else integrate polynomially. Finally, we study the action of $GL(V)$ on the integrable loci and pick one element from each orbit to list the isomorphism classes of deformations.

3. DEFORMATIONS OF THE STATIC KINEMATICAL LIE ALGEBRA

We are interested in classifying kinematical Lie algebras as deformations of the static kinematical Lie algebra \mathfrak{g} with basis R_i, B_i, P_i, H and nonzero Lie brackets:

$$[R, R] = R \quad [R, B] = B \quad \text{and} \quad [R, P] = P \quad (25)$$

in the abbreviated notation, where $[R_i, R_j] = \epsilon_{ijk}R_k$, et cetera. We let $\mathfrak{s} \cong \mathfrak{so}(3)$ denote Lie subalgebra generated by the R_i and \mathfrak{h} denote the abelian ideal spanned by B_i, P_i, H . The deformation complex $C^\bullet(\mathfrak{g}; \mathfrak{g})$ is quasi-isomorphic, by the Hochschild–Serre theorem, to the subcomplex $C^\bullet(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ of \mathfrak{s} -invariant \mathfrak{h} -cochains with values in the representation \mathfrak{g} . This has the consequence that *any* deformation of the static kinematical Lie algebra is necessarily a kinematical Lie algebra, something which was observed already in [4] in the context of galilean deformations. Because of this fact, we will work with the complex $C^\bullet := C^\bullet(\mathfrak{h}; \mathfrak{g})^\mathfrak{s}$ throughout. Let β^i, π^i, η denote the basis for \mathfrak{h}^* canonically dual to B_i, P_i, H , respectively. The Chevalley–Eilenberg differential $\partial : C^p \rightarrow C^{p+1}$ defines subspaces $Z^p \subset C^p$ of cocycles and $B^p \subset C^p$ of coboundaries. The cohomology $H^p = Z^p/B^p$ is not a subspace of C^p , but we may identify it with a subspace of C^p by making a choice of cocycle representative for each element in a basis of H^p . Let $\mathcal{H}^p \subset C^p$ be such a choice, so that we may decompose $Z^p = B^p \oplus \mathcal{H}^p$. A convenient choice is one where \mathcal{H}^p is stable under those automorphisms of \mathfrak{h} which commute with the action of \mathfrak{s} .

3.1. Automorphisms of \mathfrak{h} . Since \mathfrak{h} is abelian, the automorphism group is the general linear group $GL(\mathfrak{h})$. However we are only interested in those automorphisms which commute with the action of \mathfrak{s} , so that they act on the \mathfrak{s} -invariant deformation complex. To this end we will consider the group $G = \mathbb{R}^\times \times GL(\mathbb{R}^2)$

acting on \mathfrak{h} in the following way. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}^\times$, then

$$(\mathbf{B}, \mathbf{P}, H) \mapsto (\mathbf{B}, \mathbf{P}, H) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \lambda \end{pmatrix} = (a\mathbf{B} + c\mathbf{P}, b\mathbf{B} + d\mathbf{P}, \lambda H). \quad (26)$$

The induced action on \mathfrak{h}^* is

$$\beta \mapsto \Delta^{-1}(d\beta - b\pi) \quad \pi \mapsto \Delta^{-1}(-c\beta + a\pi) \quad \text{and} \quad \eta \mapsto \lambda^{-1}\eta, \quad (27)$$

where $\Delta = \det A = ad - bc$.

3.2. Infinitesimal deformations. In the notation of Appendix B and in particular from the action of the Chevalley–Eilenberg differential given by equation (154), we see that the spaces of coboundaries B^2 and cocycles Z^2 are given by

$$B^2 = \mathbb{R} \langle 2c_9 + c_{13}, c_{12} + 2c_{16} \rangle \quad \text{and} \quad Z^2 = \mathbb{R} \langle c_1, c_3, c_4, c_6, c_7, c_9, c_{10}, c_{12}, c_{13}, c_{15}, c_{16} \rangle, \quad (28)$$

where the notation $\mathbb{R} \langle \dots \rangle$ means the real subspace spanned by the vectors inside the angle brackets. Under the G -action, Z^2 decomposes into the following submodules:

$$Z^2 = \mathbb{R} \langle c_1 \rangle \oplus \mathbb{R} \langle c_3 + c_7 \rangle \oplus \mathbb{R} \langle c_3 - c_7, c_4, c_6 \rangle \oplus \mathbb{R} \langle c_9 - c_{13}, c_{10}, c_{12} - c_{16}, c_{15} \rangle \oplus B^2, \quad (29)$$

so that the cohomology H^2 is isomorphic (as a G -module) to the subspace $\mathcal{H}^2 \subset Z^2$ defined by

$$\mathcal{H}^2 = \mathbb{R} \langle c_1 \rangle \oplus \mathbb{R} \langle c_3 + c_7 \rangle \oplus \mathbb{R} \langle c_3 - c_7, c_4, c_6 \rangle \oplus \mathbb{R} \langle c_9 - c_{13}, c_{10}, c_{12} - c_{16}, c_{15} \rangle. \quad (30)$$

Therefore the most general (nontrivial) infinitesimal deformation can be parametrised as

$$\varphi_1 = t_1 c_1 + t_2 (c_3 + c_7) + t_3 c_4 + t_4 c_6 + t_5 (c_3 - c_7) + t_6 c_{10} + t_7 (c_{12} - c_{16}) + t_8 (c_9 - c_{13}) + t_9 c_{15}. \quad (31)$$

3.3. Obstructions. The first obstruction is the class of $\frac{1}{2}[\varphi_1, \varphi_1]$ in H^3 . Using the notation in Appendix B and in particular the determination of the Nijenhuis–Richardson bracket (156), we calculate

$$\begin{aligned} \frac{1}{2}[\varphi_1, \varphi_1] &= b_1(t_1 t_3 + 2t_6 t_7 + 2t_8^2) + b_2(-\frac{1}{2}t_4 t_6 + t_3 t_7 + \frac{3}{2}t_2 t_8 + \frac{3}{2}t_5 t_8) \\ &\quad + b_3(t_4 t_6 - 2t_3 t_7 - t_2 t_8 - t_5 t_8) + b_4(t_2 t_7 - t_5 t_7 + 2t_4 t_8 + t_3 t_9) \\ &\quad + \frac{1}{2}b_5(-t_2 t_7 + t_5 t_7 - 2t_4 t_8 - t_3 t_9) + \frac{1}{2}b_6(t_2 t_6 + 3t_5 t_6 - t_3 t_8) \\ &\quad + b_8(t_1 t_2 - t_1 t_5 + t_7 t_8 + t_6 t_9) + b_9(t_1 t_2 + t_1 t_5 - t_7 t_8 - t_6 t_9) \\ &\quad + b_{11}(t_1 t_4 - 2t_2^2 + 2t_8 t_9) + b_{12}(-\frac{1}{2}t_1 t_6) + 2b_{13}t_1 t_2 + \frac{1}{2}b_{14}(3t_4 t_7 + t_2 t_9 - 3t_5 t_9) \\ &\quad + b_{18}(-\frac{1}{2}t_1 t_8) + b_{19}(t_1 t_7) + b_{20}(\frac{1}{2}t_1 t_7 + t_1 t_9). \end{aligned} \quad (32)$$

We know that this is a cocycle in Z^3 and from equation (154) we know that B^3 is spanned by $b_1, b_2 + b_3, b_4 + b_5, b_8 - b_9, b_{11}$. In other words, if we let $[-]$ denote the class in H^3 of a cocycle, we see that

$$\begin{aligned} [\frac{1}{2}[\varphi_1, \varphi_1]] &= [b_2](-\frac{3}{2}t_4 t_6 + 3t_3 t_7 + \frac{5}{2}t_2 t_8 + \frac{5}{2}t_5 t_8) + [b_4](\frac{3}{2}t_2 t_7 - \frac{3}{2}t_5 t_7 + 3t_4 t_8 + \frac{3}{2}t_3 t_9) \\ &\quad + \frac{1}{2}[b_6](t_2 t_6 + 3t_5 t_6 - t_3 t_8) + 2[b_8]t_1 t_2 - \frac{1}{2}[b_{12}]t_1 t_6 + 2[b_{13}]t_1 t_2 \\ &\quad + \frac{1}{2}[b_{14}](3t_4 t_7 + t_2 t_9 - 3t_5 t_9) - \frac{1}{2}[b_{18}]t_1 t_8 + [b_{19}]t_1 t_7 + [b_{20}](\frac{1}{2}t_1 t_7 + t_1 t_9). \end{aligned} \quad (33)$$

This obstruction class is zero on the intersection of the following 9 quadrics²

$$\begin{aligned} t_1 t_2 &= 0 & -3t_4 t_6 + 6t_3 t_7 + 5t_2 t_8 + 5t_5 t_8 &= 0 \\ t_1 t_6 &= 0 & t_2 t_7 - t_5 t_7 + 2t_4 t_8 + t_3 t_9 &= 0 \\ t_1 t_7 &= 0 & t_2 t_6 + 3t_5 t_6 - t_3 t_8 &= 0 \\ t_1 t_8 &= 0 & 3t_4 t_7 + t_2 t_9 - 3t_5 t_9 &= 0. \\ t_1 t_9 &= 0 & & \end{aligned} \quad (34)$$

Provided that these quadratic equations are satisfied, we find that $\frac{1}{2}[\varphi_1, \varphi_1] = \partial\varphi_2$, where

$$\varphi_2 = \frac{2}{5}(2t_3 t_7 - t_4 t_6)c_2 + (t_1 t_3 + 2t_6 t_7 + 2t_8^2)c_8 + (t_1 t_2 - t_1 t_5 + t_7 t_8 + t_6 t_9)c_{11} - (t_1 t_4 - 2t_7^2 + 2t_8 t_9)c_{14}. \quad (35)$$

²A Gröbner basis for the ideal of $\mathbb{Q}[t_1, \dots, t_9]$ generated by this system of quadrics has 17 polynomials of degrees ranging from 2 to 5. Although it is possible to solve these polynomial equations and find all branches of their zero locus, we prefer a less black-boxy approach.

The next obstruction is $[\varphi_1, \varphi_2]$. Provided that the quadratic obstructions (34) are satisfied, it is given by

$$\begin{aligned} [\varphi_1, \varphi_2] = & b_7(-t_4 t_6 t_8) + \frac{1}{2} b_{10}(6t_4 t_6 t_7 + 5t_2 t_7 t_8 - 5t_5 t_7 t_8 + 10t_4 t_8^2 + 3t_2 t_6 t_9 - 3t_5 t_6 t_9) \\ & + \frac{1}{5} b_{17}(12t_2 t_7^2 - 12t_5 t_7^2 + 9t_4 t_7 t_8 + 6t_4 t_6 t_9 - 10t_2 t_8 t_9 + 10t_5 t_8 t_9). \end{aligned} \quad (36)$$

The class in H^3 corresponding to this cocycle vanishes if and only if each of the coefficients of b_7 , b_{10} , and b_{17} vanish, yielding the following system of cubics:

$$\begin{aligned} t_4 t_6 t_8 &= 0 \\ 2t_4(3t_6 t_7 + 5t_8^2) + (t_2 - t_5)(5t_7 t_8 + 3t_6 t_9) &= 0 \\ 3t_4(3t_7 t_8 + 2t_6 t_9) + 2(t_2 - t_5)(6t_7^2 - 5t_8 t_9) &= 0. \end{aligned} \quad (37)$$

If this is the case, $[\varphi_1, \varphi_2]$ vanishes (on the nose and not just in cohomology), so we can take $\varphi_3 = 0$. The next obstruction is $\frac{1}{2}[\varphi_2, \varphi_2]$, which also vanishes, since from (156) we see that c_2, c_8, c_{11}, c_{14} are contained in an abelian subalgebra of the Nijenhuis–Richardson superalgebra. Therefore we can take $\varphi_4 = 0$ as well. There are thus no further obstructions and therefore the infinitesimal deformation is integrable on the combined locus of the system (34) of quadrics and the system (37) of cubics. We could hit this system with the Gröbner hammer, but we prefer to exploit the action of the automorphisms in order to solve it in a more transparent fashion.

3.4. The action of automorphisms on the deformation parameters. The group $G = \mathbb{R}^\times \times \mathrm{GL}(2, \mathbb{R})$ acts on the nine-dimensional vector space \mathcal{H}^2 and in particular this induces an action on the coordinates t_1, \dots, t_9 which parametrise it. Let us focus on the three-dimensional subspace spanned by t_3, t_4, t_5 . Then the action of $(\lambda, A) \in G$ on (t_3, t_4, t_5) is

$$\begin{pmatrix} t_3 \\ t_4 \\ t_5 \end{pmatrix} \mapsto \frac{1}{\lambda \Delta} \begin{pmatrix} d^2 & -c^2 & 2cd \\ -b^2 & a^2 & -2ab \\ bd & -ac & ad + bc \end{pmatrix} \begin{pmatrix} t_3 \\ t_4 \\ t_5 \end{pmatrix} = \frac{1}{\lambda} M_A \begin{pmatrix} t_3 \\ t_4 \\ t_5 \end{pmatrix}, \quad (38)$$

which defines M_A and where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta = \det A$. The kernel of this three-dimensional representation of G consists of $\{(1, \mu \mathbb{1}) \mid \mu \in \mathbb{R}^\times\}$, so the representation factors via $\mathbb{R}^\times \times \mathrm{PSL}(\mathbb{R}^2)$. This representation is conformal: the matrix M_A preserves the lorentzian inner product defined by

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad (39)$$

that is, $M_A^T K M_A = K$ and hence $\lambda^{-1} M_A^T K \lambda^{-1} M_A = \lambda^{-2} K$. Therefore G is acting by Lorentz transformations and an overall scale (which can be any nonzero number). The causal type of the vector is an invariant, but then by using the rescaling we can bring the vector to one of four canonical forms: the zero vector and a choice of spacelike, timelike and null vector relative to K . We may label these orbits by choosing a representative vector $\mathbf{t} = (t_3, t_4, t_5)$ for each:

- (1) the **zero** orbit, where $\mathbf{t} = (0, 0, 0)$;
- (2) the **spacelike** orbit, where $\mathbf{t} = (0, 0, 1)$;
- (3) the **timelike** orbit, where $\mathbf{t} = (1, -1, 0)$; and
- (4) the **lightlike** orbit, where $\mathbf{t} = (1, 0, 0)$.

This gives four branches of solutions which we will study in turn.

3.5. Zero branch deformations. Here $t_3 = t_4 = t_5 = 0$, so that the system (34) of quadrics becomes

$$\begin{aligned} t_1 t_2 &= 0 & t_1 t_7 &= 0 & t_2 t_7 &= 0 \\ t_1 t_6 &= 0 & t_1 t_8 &= 0 & t_2 t_8 &= 0 \\ t_2 t_6 &= 0 & t_1 t_9 &= 0 & t_2 t_9 &= 0 \end{aligned} \quad (40)$$

and the system (37) of cubics is then identically satisfied. The deformation is given by

$$\begin{aligned} \varphi_1 &= t_1 c_1 + t_2(c_3 + c_7) + t_6 c_{10} + t_7(c_{12} - c_{16}) + t_8(c_9 - c_{13}) + t_9 c_{15} \\ \varphi_2 &= 2(t_6 t_7 + t_8^2) c_8 + (t_7 t_8 + t_6 t_9) c_{11} + 2(t_7^2 - t_8 t_9) c_{14}. \end{aligned} \quad (41)$$

3.5.1. $t_1 \neq 0$ *subbranch*. If $t_1 \neq 0$, then $t_2 = t_6 = t_7 = t_8 = t_9 = 0$, so that

$$\varphi_1 = t_1 c_1 \quad \text{and} \quad \varphi_2 = 0. \quad (42)$$

Since $t_1 \neq 0$, we can change basis so that $t_1 = 1$ and end up with the following Lie bracket (in addition to the ones involving R_i):

$$[B_i, P_j] = \delta_{ij} H, \quad (43)$$

which defines the **Carroll** algebra.

3.5.2. $t_2 \neq 0$ *subbranch*. If $t_2 \neq 0$, then $t_1 = t_6 = t_7 = t_8 = t_9 = 0$, so that

$$\varphi_1 = t_2(c_3 + c_7) \quad \text{and} \quad \varphi_2 = 0. \quad (44)$$

Since $t_2 \neq 0$, we can change basis so that $t_2 = 1$ and end up with

$$[H, B_i] = B_i \quad \text{and} \quad [H, P_i] = P_i. \quad (45)$$

3.5.3. $t_1 = t_2 = 0$ *subbranch*. If $t_1 = t_2 = 0$, we are left with four parameters t_6, t_7, t_8, t_9 defining the deformation

$$\begin{aligned} \varphi_1 &= t_6 c_{10} + t_7(c_{12} - c_{16}) + t_8(c_9 - c_{13}) + t_9 c_{15} \\ \varphi_2 &= 2(t_6 t_7 + t_8^2) c_8 + (t_7 t_8 + t_6 t_9) c_{11} + 2(t_7^2 - t_8 t_9) c_{14}. \end{aligned} \quad (46)$$

We still have the freedom to transform the parameters by the action of G . The four remaining parameters transform according to a (conformally symplectic) rational representation of $GL(\mathbb{R}^2)$:

$$\begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix} \mapsto \rho(A) \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix} := \frac{1}{\Delta^2} \begin{pmatrix} d^3 & -3c^2 d & 3cd^2 & c^3 \\ -b^2 d & a(ad + 2bc) & -b(2ad + bc) & -a^2 c \\ bd^2 & -c(2ad + bc) & d(ad + 2bc) & ac^2 \\ b^3 & -3a^2 b & 3ab^2 & a^3 \end{pmatrix} \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix}, \quad (47)$$

where $GL(\mathbb{R}^2) \ni A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta = \det A$. The representation $A \mapsto \rho(A)$ satisfies

$$\rho(A)^T \Omega \rho(A) = \frac{1}{\Delta} \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (48)$$

This representation is dual to the natural representation of $GL(\mathbb{R}^2)$ on the space of binary cubics of the form

$$t_9 X^3 - t_7 X^2 Y + t_8 X Y^2 + t_6 Y^3, \quad (49)$$

induced from the natural two-dimensional representation of $GL(\mathbb{R}^2)$ on (X, Y) . Classical invariant theory (see, e.g., [9, p.28]) tells us that there are four $GL(\mathbb{R}^2)$ -orbits in the space of (nonzero) real binary cubics, characterised by whether the corresponding binary cubics have three distinct real roots, three distinct roots (only one real), a double root or a triple root. Choosing a representative from each orbit, we have the following values of $\mathbf{t} = (t_6, t_7, t_8, t_9)$:

- (1) Three real roots: $\mathbf{t} = (0, 0, 1, 1)$. In this case, and after $\mathbf{B} \mapsto \frac{1}{3}(\mathbf{B} + \mathbf{R})$ and $\mathbf{P} \mapsto \frac{1}{\sqrt{3}}\mathbf{P}$, we arrive at

$$[B_i, B_j] = \epsilon_{ijk} B_k \quad \text{and} \quad [P_i, P_j] = \epsilon_{ijk} (B_k - R_k). \quad (50)$$

- (2) Three distinct roots, but only one real root: $\mathbf{t} = (0, 0, 1, -1)$. In this case, and after the same change of basis as in the previous case, we arrive at

$$[B_i, B_j] = \epsilon_{ijk} B_k \quad \text{and} \quad [P_i, P_j] = -\epsilon_{ijk} (B_k - R_k). \quad (51)$$

- (3) A double root: $\mathbf{t} = (0, 1, 0, 0)$. In this case, and after $\mathbf{P} \mapsto -\frac{1}{3}(\mathbf{P} - \mathbf{R})$, we arrive at

$$[P_i, P_j] = \epsilon_{ijk} P_k. \quad (52)$$

- (4) A triple root: $\mathbf{t} = (0, 0, 0, 1)$. This is simply

$$[P_i, P_j] = \epsilon_{ijk} B_k. \quad (53)$$

3.6. Spacelike branch deformations. Here $t_3 = t_4 = 0$ and $t_5 = 1$. The deformations take the form

$$\begin{aligned}\varphi_1 &= t_1 c_1 + (t_2 + 1)c_3 + (t_2 - 1)c_7 + t_6 c_{10} + t_7(c_{12} - c_{16}) + t_8(c_9 - c_{13}) + t_9 c_{15} \\ \varphi_2 &= 2(t_6 t_7 + t_8^2)c_8 + (t_1(t_2 - 1) + t_7 t_8 + t_6 t_9)c_{11} + 2(t_7^2 - t_8 t_9)c_{14},\end{aligned}\quad (54)$$

subject to the following systems of quadrics (the cubics become quadrics in this case):

$$\begin{array}{lll}t_1 t_2 = 0 & t_1 t_6 = 0 & (t_2 + 3)t_6 = 0 \\ t_6 t_9 = 0 & t_1 t_7 = 0 & (t_2 - 1)t_7 = 0 \\ t_8 t_9 = 0 & t_1 t_8 = 0 & (t_2 + 1)t_8 = 0 \\ & t_1 t_9 = 0 & (t_2 - 3)t_9 = 0\end{array}\quad (55)$$

This system breaks up into several subbranches.

3.6.1. $t_1 \neq 0$ subbranch. Here $t_1 \neq 0$, so that $t_2 = t_6 = t_7 = t_8 = t_9 = 0$. The deformation has

$$\varphi_1 = t_1 c_1 + c_3 - c_7 \quad \text{and} \quad \varphi_2 = -t_1 c_{11} \quad (56)$$

leading (after rescaling of the generators) to

$$\boxed{[H, B_i] = B_i \quad [H, P_i] = -P_i \quad \text{and} \quad [B_i, P_j] = \delta_{ij} H - \epsilon_{ijk} R_k.} \quad (57)$$

This Lie algebra is isomorphic to $\mathfrak{so}(4, 1)$, expressed relative to a Witt (i.e., lightcone) basis, where $B_i \sim L_{-i}$, $P_i \sim L_{+i}$ and $H \sim L_{+-}$. Since it is simple, the Killing form is nondegenerate and hence it is a metric Lie algebra.

3.6.2. $t_1 = 0, t_2 \neq \pm 1, \pm 3$ subbranch. Here $t_1 = 0$ and $t_2 \neq \pm 1, \pm 3$, so that $t_6 = t_7 = t_8 = t_9 = 0$. The deformation has $\varphi_1 = (t_2 + 1)c_3 + (t_2 - 1)c_7$, leading to

$$[H, B_i] = (t_2 + 1)B_i \quad \text{and} \quad [H, P_i] = (t_2 - 1)P_i. \quad (58)$$

The parameter t_2 can be further restricted by noticing that since $t_2 \neq -1$, we can rescale $H \mapsto \frac{1}{t_2 - 1}H$, so that

$$\boxed{[H, B_i] = \gamma B_i \quad \text{and} \quad [H, P_i] = P_i,} \quad (59)$$

where we have introduced $\gamma := \frac{t_2 + 1}{t_2 - 1}$. By exchanging $B \leftrightarrow P$ if necessary, we can arrange so that $\gamma \in [-1, 1)$, the case $\gamma = -1$ corresponding to the lorentzian **Newton** algebra.

3.6.3. $t_1 = 0$ and $t_2 = 1$ subbranch. Here $t_1 = 0$ and $t_2 = 1$ and hence $t_6 = t_8 = t_9 = 0$. The deformation has

$$\varphi_1 = 2c_3 + t_7(c_{12} - c_{16}) \quad \text{and} \quad \varphi_2 = 2t_7^2 c_8. \quad (60)$$

Redefining generators ($P \mapsto P - t_7 R$, $H \mapsto \frac{1}{2}H, \dots$) we are left with two possible Lie algebras depending on whether or not $t_7 = 0$:

$$\boxed{[H, B_i] = B_i,} \quad (61)$$

which is the case $t_2 = 1$ of (59), and

$$\boxed{[H, B_i] = B_i \quad \text{and} \quad [P_i, P_j] = \epsilon_{ijk} P_k.} \quad (62)$$

3.6.4. $t_1 = 0$ and $t_2 = -1$ subbranch. Here $t_1 = 0$ and $t_2 = -1$, so that $t_6 = t_7 = t_9 = 0$. This is isomorphic to the previous case under $B \leftrightarrow P$.

3.6.5. $t_1 = 0$ and $t_2 = 3$ subbranch. Here $t_1 = 0$ and $t_2 = 3$, so that $t_6 = t_7 = t_8 = 0$. The deformation has $\varphi_1 = 4c_3 + 2c_7 + t_9 c_{15}$. Rescaling H and, if $t_9 \neq 0$, also B appropriately, we may bring the Lie bracket to the following forms:

$$\boxed{[H, B_i] = 2B_i \quad \text{and} \quad [H, P_i] = P_i,} \quad (63)$$

if $t_9 = 0$, which is isomorphic to the case $t_2 = 3$ of (59), and

$$\boxed{[H, B_i] = 2B_i \quad [H, P_i] = P_i \quad \text{and} \quad [P_i, P_j] = \epsilon_{ijk} B_k,} \quad (64)$$

if $t_9 \neq 0$.

3.6.6. $t_1 = 0$ and $t_2 = -3$ subbranch. This is equivalent to the previous case under $B \leftrightarrow P$.

3.7. Timelike branch deformations. Here $t_2 = t_5$, $t_3 = -1$ and $t_4 = 1$. The system (34) of quadrics becomes

$$\begin{aligned} t_1 t_2 &= 0 & t_2 t_6 &= t_8 \\ t_1 t_6 &= 0 & t_2 t_7 &= 2t_8 - t_9 \\ t_1 t_7 &= 0 & t_2 t_8 &= -\frac{3}{5}t_6 - \frac{6}{5}t_7 \\ t_1 t_8 &= 0 & t_2 t_9 &= 3t_7. \\ t_1 t_9 &= 0 \end{aligned} \tag{65}$$

The four equations on the right can be written in the following suggestive form:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ -\frac{3}{5} & -\frac{6}{5} & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix} = t_2 \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix}, \tag{66}$$

which says that (t_6, t_7, t_8, t_9) is an eigenvector of the matrix on the LHS with (real) eigenvalue t_2 and therefore also an eigenvector of the square of that matrix with non-negative eigenvalue t_2^2 :

$$\begin{pmatrix} -\frac{3}{5} & -\frac{6}{5} & 0 & 0 \\ -\frac{6}{5} & -\frac{27}{5} & 0 & 0 \\ 0 & 0 & -3 & \frac{6}{5} \\ 0 & 0 & 6 & -3 \end{pmatrix} \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix} = t_2^2 \begin{pmatrix} t_6 \\ t_7 \\ t_8 \\ t_9 \end{pmatrix}. \tag{67}$$

Notice, however, that the above matrix is diagonalisable (over the reals) with *negative* eigenvalues $-\frac{3}{5}(5 \pm 2\sqrt{5})$, each with multiplicity 2. Therefore the above equation has as unique solution $t_6 = t_7 = t_8 = t_9 = 0$, which automatically solves the cubic system (37) and leaves the following $t_1 t_2 = 0$, $t_3 = 1$ and $t_4 = -1$. This gives rise to two branches depending on whether or not $t_1 = 0$.

3.7.1. $t_1 \neq 0$ subbranch. If $t_1 \neq 0$, then $t_2 = 0$ and the deformation has

$$\varphi_1 = t_1 c_1 + c_4 - c_6 \quad \text{and} \quad \varphi_2 = t_1 c_8 - t_1 c_{14}, \tag{68}$$

which results in the following Lie brackets:

$$\begin{aligned} [H, B_i] &= P_i & [B_i, B_j] &= t_1 \epsilon_{ijk} R_k \\ [H, P_i] &= -B_i & [P_i, P_j] &= t_1 \epsilon_{ijk} R_k \\ [B_i, P_j] &= t_1 \delta_{ij} H \end{aligned} \tag{69}$$

We may rescale \mathbf{B} and \mathbf{P} to set $t_1 = \pm 1$, depending on its sign, leading to the a Lie algebra isomorphic to one of the following two:

$$\begin{aligned} [B_i, P_j] &= \pm \delta_{ij} H \\ [H, B_i] &= P_i \\ [H, P_i] &= -B_i \end{aligned} \quad \begin{aligned} [B_i, B_j] &= \pm \epsilon_{ijk} R_k \\ [P_i, P_j] &= \pm \epsilon_{ijk} R_k, \end{aligned}$$

(70)

which is isomorphic either to $\mathfrak{so}(5)$ or to $\mathfrak{so}(3, 2)$, depending on the sign. Since these Lie algebras are simple, they are metric relative to the Killing form.

3.7.2. $t_1 = 0$ subbranch. Here the deformation is

$$\varphi_1 = t_2(c_3 + c_7) + c_4 - c_6 \tag{71}$$

which leads to the following Lie algebra

$$[H, B_i] = t_2 B_i + P_i \quad \text{and} \quad [H, P_i] = t_2 P_i - B_i,$$

(72)

where we can always arrange $t_2 \geq 0$ by relabelling generators. If $t_2 = 0$ this is the euclidean **Newton** algebra.

3.8. Lightlike branch deformations. Here $t_3 = 1$ and $t_4 = t_5 = 0$. The system (34) of quadrics becomes

$$\begin{aligned} t_1 t_2 &= 0 & t_1 t_8 &= 0 & t_7 &= \frac{5}{6} t_2 t_8 \\ t_1 t_6 &= 0 & t_1 t_9 &= 0 & t_8 &= -t_2 t_6 \\ t_1 t_7 &= 0 & t_2 t_9 &= 0 & t_9 &= t_2 t_7. \end{aligned} \quad (73)$$

Plugging $t_9 = t_2 t_7$ into $t_2 t_9 = 0$, yields $t_2^2 t_7 = 0$ which implies that $t_2 t_7 = 0$, so that $t_9 = 0$. Similarly, $t_2 t_7 = 0$ implies that $t_2 t_8 = 0$ and hence that $t_7 = 0$, and finally this implies that $t_8 = 0$ as well. This already means that the cubic system (37) is identically satisfied and when the dust clears we are left with $t_3 = 1$, $t_4 = t_5 = t_7 = t_8 = t_9 = 0$, subject to the following conditions:

$$t_1 t_2 = 0 \quad t_1 t_6 = 0 \quad \text{and} \quad t_2 t_6 = 0. \quad (74)$$

This gives rise to three branches of solutions.

3.8.1. $t_1 \neq 0$ subbranch. In this case, $t_1 \neq 0$ and hence $t_2 = t_6 = 0$. The deformation is

$$\varphi_1 = t_1 c_1 + c_4 \quad \text{and} \quad \varphi_2 = t_1 c_8, \quad (75)$$

resulting in the following Lie brackets:

$$[H, B_i] = P_i \quad [B_i, P_j] = t_1 \delta_{ij} H \quad \text{and} \quad [B_i, B_j] = t_1 \epsilon_{ijk} R_k. \quad (76)$$

We may rescale the generators in such a way that we reabsorb t_1 up to its sign and arrive at two non-isomorphic deformations

$$\boxed{[H, B_i] = \pm P_i \quad [B_i, P_j] = \delta_{ij} H \quad \text{and} \quad [B_i, B_j] = \pm \epsilon_{ijk} R_k,} \quad (77)$$

which correspond to the **euclidean** \mathfrak{e} and **Poincaré** \mathfrak{p} Lie algebras.

3.8.2. $t_2 \neq 0$ subbranch. Here $t_2 \neq 0$, so $t_1 = t_6 = 0$. The deformation is therefore

$$\varphi_1 = t_2(c_3 + c_7) - c_4, \quad (78)$$

which after rescaling generators can be brought to the form

$$\boxed{[H, B_i] = B_i + P_i \quad \text{and} \quad [H, P_i] = P_i.} \quad (79)$$

3.8.3. $t_1 = t_2 = 0$ subbranch. Here $t_1 = t_2 = 0$ and hence the deformation is given by

$$\varphi_1 = c_4 + t_6 c_{10}, \quad (80)$$

leading to the Lie brackets

$$[H, B_i] = P_i \quad \text{and} \quad [B_i, B_j] = t_6 \epsilon_{ijk} P_k. \quad (81)$$

If $t_6 = 0$, we arrive at the **galilean** algebra (after rescaling)

$$\boxed{[H, B_i] = -P_i.} \quad (82)$$

If $t_6 \neq 0$, we can rescale the generators to arrive at

$$\boxed{[H, B_i] = -P_i \quad \text{and} \quad [B_i, B_j] = \epsilon_{ijk} P_k.} \quad (83)$$

3.9. Invariant inner products. We shall now analyse the existence of invariant inner products on the Lie algebras determined in this section. Recall that an invariant inner product on a Lie algebra \mathfrak{g} is a nondegenerate symmetric bilinear form $(-, -) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ which is “associative”; that is,

$$([x, y], z) = (x, [y, z]) \quad \forall x, y, z \in \mathfrak{g}. \quad (84)$$

The Killing form is associative, but it is only nondegenerate for semisimple Lie algebras, so that the inner product on non-semisimple metric Lie algebras is always an additional piece of data. When it exists, it is seldom positive-definite, unless \mathfrak{g} is the Lie algebra of a compact group. This means that it is the direct sum of a semisimple Lie algebra (of compact type) and an abelian Lie algebra.

Rather than appealing to any general structural results, the strategy here is simply to exploit the associativity condition (84). We shall first of all show that no kinematical Lie algebra where \mathbf{B} and \mathbf{P} span an abelian ideal can be metric. This will rule out the first eight cases in Table 1. Indeed, let $(-, -)$ be an associative symmetric bilinear form. We will show that it is degenerate. To this end, let X, Y be any of \mathbf{B}, \mathbf{P} and consider

$$\epsilon_{ijk}(X_k, Y_\ell) = ([R_i, X_j], Y_\ell) = (R_i, [X_j, Y_\ell]) = 0, \quad (85)$$

where we have used associativity and the fact that X, Y are vectors under rotations. Therefore the only nonzero components of $(-, -)$ are

$$(H, H) \quad (R_i, R_j) \quad (R_i, B_j) \quad (R_i, P_j) \quad (86)$$

and hence there is some nonzero $Z_i = \alpha B_i + \beta P_i$, for some $\alpha, \beta \in \mathbb{R}$ (not both zero), which obeys $(Z_i, -) = 0$.

Any associative symmetric bilinear form in the Carroll algebra is degenerate, since $(H, -) = 0$. Indeed, by rotational invariance, the only possible nonzero inner product of H is with itself, but then

$$\delta_{ij}(H, H) = (H, [B_i, P_j]) = ([H, B_i], P_j) = 0. \quad (87)$$

The simple Lie algebras $\mathfrak{so}(4, 1)$, $\mathfrak{so}(5)$ and $\mathfrak{so}(3, 2)$ are of course metric relative to the Killing form, whereas the euclidean and Poincaré algebras (in this dimension) are not metric. Indeed, let $(-, -)$ be an associative symmetric bilinear form on either \mathfrak{e} or \mathfrak{p} and calculate (H, H) , which is the only possibly nonzero rotationally invariant inner product involving H :

$$\delta_{ij}(H, H) = (H, [B_i, P_j]) = -(H, [P_j, B_i]) = -([H, P_j], B_i) = 0. \quad (88)$$

This settles all the Lie algebras above the line in Table 1. Of the seven Lie algebra below the line in that table, it will turn out that the first four are metric, but not the last three. Let's do them first.

Consider the Lie algebra in (62) and let $(-, -)$ be an associative symmetric bilinear form. If X is any one of R, B , then

$$(P_i, X_j) = ([H, P_i], X_j) = -([P_i, H], X_j) = -(P_i, [H, X_j]) = 0, \quad (89)$$

whereas

$$(P_i, P_j) = ([H, P_i], P_j) = (H, [P_i, P_j]) = 0. \quad (90)$$

Therefore, $(P_i, -) = 0$.

Let $(-, -)$ be an associative symmetric bilinear form on the Lie algebra in (83). Then again if X is any of R, P ,

$$(P_i, X_j) = ([B_i, H], X_j) = (B_i, [H, X_j]) = 0, \quad (91)$$

whereas

$$(B_i, P_j) = (B_i, [B_j, H]) = ([B_i, B_j], H) = \epsilon_{ijk}(P_k, H) = 0, \quad (92)$$

by rotational invariance. Therefore $(P_i, -) = 0$.

Let $(-, -)$ be an associative symmetric bilinear form on the Lie algebra in (64). Then if X is either B or P ,

$$(B_i, X_j) = \frac{1}{2}([H, B_i], X_j) = \frac{1}{2}(H, [B_i, X_j]) = 0, \quad (93)$$

whereas

$$(R_i, B_j) = \frac{1}{2}(R_i, [H, B_j]) = \frac{1}{2}([R_i, H], B_j) = 0, \quad (94)$$

so that $(B_i, -) = 0$.

The first four Lie algebras below the line in Table 1 are metric under a four-parameter family of associative inner products. For these algebras H remains central, so one of the parameters is (H, H) , which has to be different from zero. To describe the other three parameters, let us encode the associative inner product on the nine-dimensional subalgebra spanned by R_i, B_i, P_i as a 3×3 symmetric matrix:

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \quad \text{where} \quad (R_i, R_j) = b_{11}\delta_{ij}, (R_i, B_j) = b_{12}\delta_{ij}, \dots, (P_i, P_j) = b_{33}\delta_{ij}; \quad (95)$$

although it is important to keep in mind that the nondegeneracy of the inner product is *not equivalent* to the nondegeneracy of this symmetric matrix. (It is the trace, not the determinant, which is multiplicative over the tensor product.) We will simply list the matrices for each of the Lie algebras in question, along with the condition of nondegeneracy on the parameters.

For the Lie algebra in (50), we have

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & 0 \\ \gamma & 0 & \beta - \alpha \end{pmatrix} \quad \beta((\alpha - \beta)^2 + \gamma^2) \neq 0. \quad (96)$$

For the Lie algebra in (51), we have

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & 0 \\ \gamma & 0 & \alpha - \beta \end{pmatrix} \quad \beta((\alpha - \beta)^2 + \gamma^2) \neq 0. \quad (97)$$

For the Lie algebra in (52), we have

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \beta & 0 \\ \gamma & 0 & 0 \end{pmatrix} \quad \beta\gamma \neq 0. \quad (98)$$

Finally, for the Lie algebra in (53), we have

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & 0 \\ \gamma & 0 & 0 \end{pmatrix} \quad \gamma \neq 0. \quad (99)$$

3.10. Summary. The classification in this section is of course not new: the Lie algebras agree precisely with the kinematical Lie algebras classified by Bacry and Nuyts in [2]. Table 1 lists our results and they can be compared with Table 1 in that paper. Our notation differs from that in [2] in that our \mathbf{R} and \mathbf{B} are their \mathbf{J} and \mathbf{K} , respectively.

All kinematical Lie algebras share the following Lie brackets (in abbreviated notation):

$$[\mathbf{R}, \mathbf{R}] = \mathbf{R} \quad [\mathbf{R}, \mathbf{B}] = \mathbf{B} \quad [\mathbf{R}, \mathbf{H}] = 0 \quad \text{and} \quad [\mathbf{R}, \mathbf{P}] = \mathbf{P}, \quad (100)$$

so in the table we will only list any additional brackets. The **static** kinematical Lie algebra has no additional nonzero brackets and is listed first, for completeness. In some cases we have relabelled generators ($\mathbf{B} \leftrightarrow \mathbf{P}$) in order to arrive at a more uniform description. It follows from the classifications of kinematical Lie algebras in dimension $D + 1$ for $D \geq 4$ [5] and for $D = 2$ [6] that the kinematical Lie algebras in Table 1 which lie below the line are unique to $D = 3$: indeed, they owe their existence to the vector product in \mathbb{R}^3 , which is invariant under rotations.

TABLE 1. Kinematical Lie algebras

Eq.	Nonzero Lie brackets				Comments	Metric?	
25					static		
82	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$				galilean		
59	$[\mathbf{H}, \mathbf{B}] = \gamma \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			$0, \frac{1}{2} \neq \gamma \in (-1, 1)$		
59	$[\mathbf{H}, \mathbf{B}] = -\mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			lorentzian Newton		
45	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			$\gamma = 1$ in (59)		
61	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			$\gamma = 0$ in (59)		
63	$[\mathbf{H}, \mathbf{B}] = 2\mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			$\gamma = \frac{1}{2}$ in (59)		
72	$[\mathbf{H}, \mathbf{B}] = \alpha \mathbf{B} + \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \alpha \mathbf{P} - \mathbf{B}$			$\alpha > 0$		
72	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$			euclidean Newton		
79	$[\mathbf{H}, \mathbf{B}] = \mathbf{B} + \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$					
43	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$				Carroll		
77	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$		ϵ (euclidean)		
77	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$		\mathfrak{p} (Poincaré)		
57	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{P}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H} - \mathbf{R}$		$\mathfrak{so}(4, 1)$	✓	
70	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R}$	✓	
70	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = -\mathbf{H}$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{R}$	✓	
50					$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{B} - \mathbf{R}$	✓
51					$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R} - \mathbf{B}$	✓
52					$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$		✓
53					$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$		✓
62	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$				$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$		
83	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$			$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$			
64	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = 2\mathbf{P}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$				

4. DEFORMATIONS OF THE CENTRALLY-EXTENDED STATIC KINEMATICAL LIE ALGEBRA

As shown in Appendix A.3, the static kinematical Lie algebra \mathfrak{g} given by (25) admits a one-dimensional universal central extension $\tilde{\mathfrak{g}}$, generated by $\mathbf{R}_i, \mathbf{B}_i, \mathbf{P}_i, \mathbf{H}, \mathbf{Z}$ and nonzero Lie brackets in abbreviated notation:

$$[\mathbf{R}, \mathbf{R}] = \mathbf{R} \quad [\mathbf{R}, \mathbf{B}] = \mathbf{B} \quad [\mathbf{R}, \mathbf{P}] = \mathbf{P} \quad \text{and} \quad [\mathbf{B}, \mathbf{P}] = \mathbf{Z}. \quad (101)$$

We will let $\tilde{\mathfrak{h}}$ denote the ideal generated by $\mathbf{B}, \mathbf{P}, \mathbf{H}, \mathbf{Z}$ and again \mathfrak{s} the rotational subalgebra generated by \mathbf{R} . By the Hochschild–Serre decomposition theorem, $H^2(\tilde{\mathfrak{g}}; \tilde{\mathfrak{g}}) \cong H^2(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$, which can be calculated by the \mathfrak{s} -invariant subcomplex $C^\bullet(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^{\mathfrak{s}}$ described in Appendix C. As in the case of the static kinematical Lie algebra \mathfrak{g} treated in Section 3, it will be convenient to exploit the action of those automorphisms of $\tilde{\mathfrak{h}}$ which commute with \mathfrak{s} .

4.1. Automorphisms of $\tilde{\mathfrak{h}}$. Let $\tilde{G} = \mathrm{GL}(\mathbb{R}^2) \ltimes \mathrm{Aff}(\Lambda^2 \mathbb{R}^2)$ denote the semidirect product of the group $\mathrm{GL}(\mathbb{R}^2)$ of invertible linear transformations of \mathbb{R}^2 and the group $\mathrm{Aff}(\Lambda^2 \mathbb{R}^2)$ of invertible affine transformations on the one-dimensional vector space $\Lambda^2 \mathbb{R}^2$. The reason we do not simply call this \mathbb{R} is that $\Lambda^2 \mathbb{R}^2$ is the one-dimensional (determinant) representation of $\mathrm{GL}(\mathbb{R}^2)$. This group acts on $\tilde{\mathfrak{h}}$ by automorphisms as follows:

$$(\mathbf{B}, \mathbf{P}, H, Z) \mapsto (\mathbf{B}, \mathbf{P}, H, Z) \begin{pmatrix} \mathbf{a} & \mathbf{b} & 0 & 0 \\ \mathbf{c} & \mathbf{d} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \mu\Delta & \Delta \end{pmatrix} = (\mathbf{a}\mathbf{B} + \mathbf{c}\mathbf{P}, \mathbf{b}\mathbf{B} + \mathbf{d}\mathbf{P}, \lambda H + \mu\Delta Z, \Delta Z), \quad (102)$$

where $\Delta = \det A = ad - bc$. The induced action on $\tilde{\mathfrak{h}}^*$ is

$$\beta \mapsto \Delta^{-1}(d\beta - b\pi) \quad \pi \mapsto \Delta^{-1}(-c\beta + a\pi) \quad \eta \mapsto \lambda^{-1}\eta \quad \text{and} \quad \zeta \mapsto \Delta^{-1}\zeta - \lambda^{-1}\mu\eta. \quad (103)$$

4.2. Infinitesimal deformations. In the notation of Appendix C and taking into account the action of the Chevalley–Eilenberg differential on \mathfrak{s} -invariant cochains given by equation (158) we see that the spaces of coboundaries B^2 and cocycles Z^2 are given by

$$B^2 = \mathbb{R} \langle \tilde{c}_1, \tilde{c}_2, \tilde{c}_4 + 2\tilde{c}_{11}, \tilde{c}_5 + 2\tilde{c}_7 \rangle \quad (104)$$

and

$$Z^2 = B^2 \oplus \mathbb{R} \langle \tilde{c}_{14} + \tilde{c}_{16} + 2\tilde{c}_{24}, \tilde{c}_{14} - \tilde{c}_{16}, \tilde{c}_{13}, \tilde{c}_3 + \tilde{c}_{20} - \tilde{c}_{22}, \tilde{c}_{17}, \tilde{c}_9 - \tilde{c}_{19}, \tilde{c}_6 + \tilde{c}_{23} \rangle, \quad (105)$$

where the chosen basis is adapted to the \tilde{G} -action. The most general (nontrivial) infinitesimal deformation is parametrised by $(u_1, \dots, u_7) \in \mathbb{R}^7$ as

$$\begin{aligned} \varphi_1 = & u_1(\tilde{c}_{14} + \tilde{c}_{16} + 2\tilde{c}_{24}) - u_2(\tilde{c}_{14} - \tilde{c}_{16}) + u_3\tilde{c}_{13} - u_4\tilde{c}_{17} \\ & + u_5(\tilde{c}_3 + \tilde{c}_{20} - \tilde{c}_{22}) + u_6(\tilde{c}_9 - \tilde{c}_{19}) + u_7(\tilde{c}_6 + \tilde{c}_{23}), \end{aligned} \quad (106)$$

where we have altered the signs of u_2 and u_4 to obtain cleaner formulae later on.

4.3. Obstructions. The first obstruction is the class of $\frac{1}{2}[\varphi_1, \varphi_1]$. Using the formulae (160) for the restriction of the Nijenhuis–Richardson bracket to the \mathfrak{s} -invariant cochains, we find that

$$\begin{aligned} \frac{1}{2}[\varphi_1, \varphi_1] = & (2u_1u_5 + u_3u_7 - u_4u_6)(\tilde{b}_6 + \tilde{b}_{22} - \tilde{b}_2) \\ & + (u_1u_7 + u_2u_7 - u_4u_5)(2\tilde{b}_3 + \tilde{b}_{19}) + (u_3u_5 - u_1u_6 + u_2u_6)(\tilde{b}_{25} - 2\tilde{b}_5). \end{aligned} \quad (107)$$

These cocycles are all nontrivial and linearly independent in cohomology, so this obstruction vanishes if and only if the cocycle vanishes. This means that the integrability locus is the solution of the system of quadrics

$$\begin{aligned} 2u_1u_5 + u_3u_7 - u_4u_6 &= 0 \\ u_1u_7 + u_2u_7 - u_4u_5 &= 0 \\ u_3u_5 - u_1u_6 + u_2u_6 &= 0. \end{aligned} \quad (108)$$

If these equations are satisfied, $[\varphi_1, \varphi_1] = 0$ so that we can take $\varphi_2 = 0$ and hence there are no further obstructions. We study the system (108) by first exploiting the action of the automorphisms in order to bring the parameters to normal forms.

4.4. The action of automorphisms on the deformation parameters. The action of \tilde{G} on the cochains induces a linear action on the parameter space, which can be described as follows:

$$\mathbf{u} = (u_1, \dots, u_7)^T \mapsto \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1}\rho(A) & \lambda^{-1}\mu\rho(A) \\ 0 & 0 & \Delta^{-1}\rho(A) \end{pmatrix} \mathbf{u}, \quad (109)$$

where $\lambda \in \mathbb{R}^\times$, $\mu \in \mathbb{R}$ and if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\mathbb{R}^2)$, $\Delta = \det A = ad - bc$ and

$$\rho(A) = \frac{1}{\Delta} \begin{pmatrix} ad + bc & -ac & -bd \\ -2ab & a^2 & b^2 \\ -2cd & c^2 & d^2 \end{pmatrix}. \quad (110)$$

The representation ρ of $\mathrm{GL}(\mathbb{R}^2)$ defined by $A \mapsto \rho(A)$ has kernel the subgroup $\{a\mathbb{1} \mid a \in \mathbb{R}^\times\}$ of scalar matrices, so that it descends to a representation of the projective linear group $\mathrm{PSL}(\mathbb{R}^2) \cong \mathrm{SO}(2, 1)_o$, the

identity component of the three-dimensional Lorentz group. Indeed, this representation preserves a lorentzian inner product on the three-dimensional space of parameters (u_5, u_6, u_7) :

$$\rho(A)^T K \rho(A) = K \quad \text{for} \quad K = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (111)$$

Since we only have at our disposal the identity component of the Lorentz group, we preserve time-orientation for causal vectors. Therefore we have the following six normal forms, with the corresponding $\mathbf{t} = (u_5, u_6, u_7)$:

- (1) the **zero** orbit, where $\mathbf{t} = (0, 0, 0)$;
- (2) the **spacelike** orbit, where $\mathbf{t} = (1, 0, 0)$;
- (3) the **future timelike** orbit, where $\mathbf{t} = (0, 1, 1)$;
- (4) the **past timelike** orbit, where $\mathbf{t} = (0, -1, -1)$;
- (5) the **future lightlike** orbit, where $\mathbf{t} = (0, 0, 1)$; and
- (6) the **past lightlike** orbit, where $\mathbf{t} = (0, 0, -1)$.

This gives six branches of solutions which we will study in turn.

4.5. Zero branch deformations. In this case $u_5 = u_6 = u_7 = 0$ and the infinitesimal deformation is already integrated:

$$\varphi_1 = u_1(\tilde{c}_{14} + \tilde{c}_{16} + 2\tilde{c}_{24}) - u_2(\tilde{c}_{14} - \tilde{c}_{16}) + u_3\tilde{c}_{13} - u_4\tilde{c}_{17}. \quad (112)$$

The additional Lie brackets are (in abbreviated form)

$$\begin{aligned} [H, \mathbf{B}] &= (u_1 + u_2)\mathbf{B} - u_4\mathbf{P} \\ [H, \mathbf{P}] &= u_3\mathbf{B} + (u_1 - u_2)\mathbf{P} \\ [H, Z] &= 2u_1Z. \end{aligned} \quad (113)$$

We must distinguish two subbranches, depending on whether or not $u_1 = 0$.

4.5.1. $u_1 = 0$ subbranch. If $u_1 = 0$, we obtain

$$\begin{aligned} [H, \mathbf{B}] &= u_2\mathbf{B} - u_4\mathbf{P} \\ [H, \mathbf{P}] &= u_3\mathbf{B} - u_2\mathbf{P}, \end{aligned} \quad (114)$$

which, depending on the sign of the discriminant $\delta := u_2^2 - u_3u_4$, is isomorphic to a (non-trivial) central extension of one of the following deformations of the static kinematical Lie algebra:

- (1) $\delta > 0$: then we can change basis so that

$$\boxed{[H, \mathbf{B}] = \mathbf{B} \quad \text{and} \quad [H, \mathbf{P}] = -\mathbf{P},} \quad (115)$$

which is isomorphic to the lorentzian Newton algebra. The corresponding deformation is the well-known universal central extension of the lorentzian Newton algebra.

- (2) $\delta < 0$: then we can change basis so that

$$\boxed{[H, \mathbf{B}] = \mathbf{P} \quad \text{and} \quad [H, \mathbf{P}] = -\mathbf{B},} \quad (116)$$

which is isomorphic to the euclidean Newton algebra. The corresponding deformation is now the well-known universal central extension of the euclidean Newton algebra.

- (3) $\delta = 0$: then we can change basis so that

$$\boxed{[H, \mathbf{B}] = -\mathbf{P},} \quad (117)$$

isomorphic to the galilean algebra. In other words, this deformation is isomorphic to the **Bargmann** algebra: the universal central extension of the galilean algebra.

4.5.2. $u_1 \neq 0$ subbranch. If $u_1 \neq 0$ then we obtain a *non-central* extension of some of the deformations of the static kinematical Lie algebra. Indeed, Z generates an ideal and quotienting by this ideal gives, depending on the values of (u_1, u_2, u_3, u_4) , one of the deformations of the static kinematical Lie algebra. The action of \tilde{G} on the subspace of parameters with $u_5 = u_6 = u_7 = 0$ can be read off from equation (109):

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \mapsto \frac{1}{\lambda\Delta} \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & ad + bc & -ac & -bd \\ 0 & -2ab & a^2 & b^2 \\ 0 & -2cd & c^2 & d^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \quad (118)$$

Taking $\lambda = u_1$, we can set $u_1 = 1$ without loss of generality. The remaining parameters transform under $GL(\mathbb{R}^2)$ as a three-dimensional vector under the identity component of the Lorentz group. The orbits are classified by their lorentzian norm $u_2^2 - u_3u_4$, which can be any real number. We obtain therefore the following isomorphism classes of deformations:

- (1) $u_2^2 - u_3u_4 > 0$:

$$\boxed{[H, B] = \gamma B \quad [H, P] = P \quad [H, Z] = (\gamma + 1)Z,} \quad (119)$$

for $\gamma \in (-1, 1]$. This Lie algebra \hat{g} is a non-central extension

$$0 \longrightarrow \mathbb{R}\langle Z \rangle \longrightarrow \hat{g} \longrightarrow \hat{h} \longrightarrow 0, \quad (120)$$

of the Lie algebra, denoted here by \hat{h} , given by equations (59) (for $\gamma \neq -1, 0, \frac{1}{2}, 1$), (45) (for $\gamma = 1$), (61) (for $\gamma = 0$) or (63) (for $\gamma = \frac{1}{2}$.) The limiting case $\gamma = -1$ is the central extension of the lorentzian Newton algebra discussed above (corresponding to $u_1 = 0$).

- (2) $u_2^2 - u_3u_4 = 0$:

$$\boxed{[H, B] = B + P \quad [H, P] = P \quad [H, Z] = 2Z.} \quad (121)$$

This Lie algebra \hat{g} is a non-central extension

$$0 \longrightarrow \mathbb{R}\langle Z \rangle \longrightarrow \hat{g} \longrightarrow \hat{h} \longrightarrow 0, \quad (122)$$

of the Lie algebra \hat{h} given by equation (79).

- (3) $u_2^2 - u_3u_4 < 0$:

$$\boxed{[H, B] = \alpha B + P \quad [H, P] = \alpha P - B \quad [H, Z] = 2\alpha Z,} \quad (123)$$

for $\alpha > 0$. This Lie algebra \hat{g} is a non-central extension

$$0 \longrightarrow \mathbb{R}\langle Z \rangle \longrightarrow \hat{g} \longrightarrow \hat{h} \longrightarrow 0, \quad (124)$$

of the Lie algebra \hat{h} given by equation (72). The limiting case $\alpha = 0$ is the central extension of the euclidean Newton algebra discussed above (corresponding to $u_1 = 0$).

4.6. Spacelike branch deformations. Here $u_5 = 1$ and $u_6 = u_7 = 0$. The system (108) of quadrics becomes $u_1 = u_3 = u_4 = 0$, so the deformation becomes

$$\varphi_1 = -u_2(\tilde{c}_{14} - \tilde{c}_{16}) + \tilde{c}_3 + \tilde{c}_{20} - \tilde{c}_{22}, \quad (125)$$

leading to the Lie brackets

$$\begin{aligned} [H, B] &= u_2 B & [Z, B] &= -B & [B, P] &= Z + R. \\ [H, P] &= -u_2 P & [Z, P] &= P \end{aligned} \quad (126)$$

It follows that $H + u_2 Z$ is central, so that this deformation is a trivial central extension of the Lie algebra

$$\boxed{[Z, B] = -B \quad [Z, P] = P \quad \text{and} \quad [B, P] = Z + R,} \quad (127)$$

which is isomorphic (with Z here playing the rôle of $-H$ there) to the Lie algebra in (57); that is, to $\mathfrak{so}(4, 1)$.

4.7. Timelike branches deformations. Let us introduce $\varepsilon = \pm 1$ and treat both branches simultaneously. Here $u_5 = 0$ and $u_6 = u_7 = \varepsilon$. The system (108) of quadrics says that $u_1 = u_2 = 0$ and that $u_3 = u_4$, so that the deformation is given by

$$\varphi_1 = u_3(\tilde{c}_{13} - \tilde{c}_{17}) + \varepsilon(\tilde{c}_6 + \tilde{c}_9 - \tilde{c}_{19} + \tilde{c}_{23}), \quad (128)$$

with Lie brackets

$$\begin{aligned} [H, B] &= -u_3 P & [Z, B] &= \varepsilon P & [B, B] &= \varepsilon R \\ [H, P] &= u_3 B & [Z, P] &= -\varepsilon B & [P, P] &= \varepsilon R. \end{aligned} \quad (129)$$

It follows that $\varepsilon H + u_3 Z$ is central, and we have a trivial central extension to the Lie algebra with brackets

$$\boxed{\begin{aligned} [Z, B] &= \varepsilon P & [B, B] &= \varepsilon R & [B, P] &= Z, \\ [Z, P] &= -\varepsilon B & [P, P] &= \varepsilon R. \end{aligned}} \quad (130)$$

which (again Z playing the rôle of εH) is isomorphic to the Lie algebra in (70); that is, to $\mathfrak{so}(5)$ or $\mathfrak{so}(3, 2)$ depending on ε .

4.8. Lightlike branches deformations. We again introduce $\varepsilon = \pm 1$ and treat both branches simultaneously. We have that $u_5 = u_6 = 0$ and $u_7 = \varepsilon$. The system (108) of quadrics imply that $u_3 = 0$ and $u_2 = -u_1$, so that the deformation ends up being

$$\varphi_1 = 2u_1(\tilde{c}_{14} + \tilde{c}_{24}) - u_4\tilde{c}_{17} + \varepsilon(\tilde{c}_6 + \tilde{c}_{23}), \quad (131)$$

with Lie brackets

$$\begin{aligned} [H, B] &= -u_4 P & [H, Z] &= 2u_1 Z & [B, B] &= \varepsilon R. \\ [H, P] &= 2u_1 P & [Z, B] &= \varepsilon P \end{aligned} \quad (132)$$

Let us change basis from (H, Z) to $(H + \varepsilon u_4 Z, Z)$. In this new basis, the nonzero brackets are

$$\begin{aligned} [H, P] &= 2u_1 P & [Z, B] &= \varepsilon P \\ [H, Z] &= 2u_1 Z & [B, B] &= \varepsilon R. \end{aligned} \quad (133)$$

We must distinguish two cases, depending on whether or not $u_1 = 0$.

4.8.1. $u_1 = 0$ subbranch. If $u_1 = 0$, then (the new) H is central and we obtain a trivial central extension of the Lie algebra with brackets

$$\boxed{[B, P] = Z \quad [Z, B] = \varepsilon P \quad \text{and} \quad [B, B] = \varepsilon R,} \quad (134)$$

which is isomorphic to either the euclidean or Poincaré Lie algebras (with Z playing the rôle of $-\varepsilon H$) depending on ε .

4.8.2. $u_1 \neq 0$ subbranch. If $u_1 \neq 0$, then we may rescale H to set $u_1 = 1$ and arrive at the Lie algebra with nonzero brackets

$$\boxed{\begin{aligned} [H, P] &= P & [B, P] &= Z & [B, B] &= \varepsilon R, \\ [H, Z] &= Z & [Z, B] &= \varepsilon P \end{aligned}} \quad (135)$$

which leads to a deformation isomorphic to either the conformal euclidean or conformal Poincaré Lie algebras, depending on the sign of ε . In other words, $\mathfrak{co}(4) \ltimes \mathbb{R}^4$ or $\mathfrak{co}(3, 1) \ltimes \mathbb{R}^{3,1}$, with Z playing the rôle of the fourth translation and H playing the rôle of the dilatation.

4.9. Invariant inner products. We shall now analyse the existence of invariant inner products on the Lie algebras determined in this section, as we did in Section 3.9 for the kinematical Lie algebras classified in Section 3. In some cases we will appeal to a general result about associative inner products on Lie algebras, which says that the center $Z(\mathfrak{g})$ of a Lie algebra with an invariant inner product is the perpendicular of the first derived ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$; that is $\mathfrak{g}' = Z(\mathfrak{g})^\perp$. Therefore if \mathfrak{g} is such that $Z(\mathfrak{g}) = 0$ and $\mathfrak{g}' \subsetneq \mathfrak{g}$, then \mathfrak{g} cannot admit an invariant inner product. This is precisely the situation of the Lie algebras in the bottom third (below the line) of Table 2.

The first Lie algebra in the table (with brackets given by (101)) does not admit an invariant inner product. Indeed, if $(-, -)$ is an associative symmetric bilinear form, it follows that

$$\delta_{ij}(Z, Z) = ([B_i, P_j], Z) = (B_i, [P_j, Z]) = 0 \quad (136)$$

and

$$\delta_{ij}(Z, H) = ([B_i, P_j], H) = (B_i, [P_j, H]) = 0, \quad (137)$$

so that $(Z, -) = 0$. The exact same calculation shows that in the Bargmann algebra (117) any associative symmetric bilinear form has $(Z, -) = 0$. A very similar argument shows that the trivial central extensions of the euclidean and Poincaré algebras (134) do not admit invariant inner products either. Indeed, if $(-, -)$ is any associative symmetric bilinear form, then

$$\delta_{ij}(H, H) = ([B_i, P_j], H) = (B_i, [P_j, H]) = 0 \quad (138)$$

and

$$\delta_{ij}(H, Z) = ([B_i, P_j], Z) = (B_i, [P_j, Z]) = 0, \quad (139)$$

so that $(H, -) = 0$. The trivial central extensions of $\mathfrak{so}(4, 1)$, $\mathfrak{so}(5)$ and $\mathfrak{so}(3, 2)$ do admit invariant inner products by taking the Killing form on the simple factor and some nonzero value for (Z, Z) .

Finally, we treat the centrally extended Newton algebras. The two cases are very similar, so we give details only for the case of the lorentzian algebra (115). Let $(-, -)$ be an associative symmetric bilinear

form. We will show that $(B_i, -) = 0$, so that it is degenerate. First of all, by rotational invariance, $(B_i, H) = (B_i, Z) = 0$. Let us calculate the others:

$$\begin{aligned} (B_i, R_j) &= ([H, B_i], R_j) = -([B_i, H], R_j) = -(B_i, [H, R_j]) = 0 \\ (B_i, B_j) &= ([H, B_i], B_j) = (H, [B_i, B_j]) = 0 \\ \epsilon_{ij\ell}(B_\ell, P_k) &= ([R_i, B_j], P_k) = (R_i, [B_j, P_k]) = \delta_{jk}(R_i, Z) = 0. \end{aligned} \quad (140)$$

The euclidean case (116) is similar. In summary, only the trivial central extensions of the simple kinematical Lie algebras $\mathfrak{so}(4, 1)$, $\mathfrak{so}(5)$ and $\mathfrak{so}(3, 2)$ admit invariant inner products.

4.10. Summary. The results of this section are partially known and partially new. They extend and in at least one case correct the results of our 1989 paper [4] on the deformations of the galilean and Bargmann algebras. Table 2 lists our results. All of these Lie algebras share the following Lie brackets (in abbreviated notation):

$$[R, R] = R \quad [R, B] = B \quad [R, P] = P \quad [R, H] = 0 \quad \text{and} \quad [R, Z] = 0. \quad (141)$$

In the table we will only list any additional nonzero brackets. In some cases we have interchanged Z and H to make the notation more uniform. The table is divided into three: the top third consists of (nontrivial) central extensions, the middle third of trivial central extensions and the bottom third of non-central extensions of kinematical Lie algebras.

TABLE 2. Deformations of the centrally extended static kinematical Lie algebra

Eq.	Nonzero Lie brackets					Comments	Metric?
101	$[B, P] = Z$					centrally extended static	
115	$[B, P] = Z$	$[H, B] = B$	$[H, P] = -P$			central extension of lorentzian Newton	
116	$[B, P] = Z$	$[H, B] = P$	$[H, P] = -B$			central extension of euclidean Newton	
117	$[B, P] = Z$	$[H, B] = -P$				Bargmann	
134	$[B, P] = H$	$[H, B] = P$		$[B, B] = R$	$\epsilon \oplus \mathbb{R}Z$		
134	$[B, P] = H$	$[H, B] = -P$		$[B, B] = -R$	$p \oplus \mathbb{R}Z$		
127	$[B, P] = H + R$	$[H, B] = -B$	$[H, P] = P$			$so(4, 1) \oplus \mathbb{R}Z$	✓
130	$[B, P] = H$	$[H, B] = P$	$[H, P] = -B$	$[P, P] = R$	$[B, B] = R$	$so(5) \oplus \mathbb{R}Z$	✓
130	$[B, P] = H$	$[H, B] = -P$	$[H, P] = B$	$[P, P] = -R$	$[B, B] = -R$	$so(3, 2) \oplus \mathbb{R}Z$	✓
119	$[B, P] = Z$	$[H, B] = B$	$[H, P] = P$	$[H, Z] = 2Z$			
119	$[B, P] = Z$	$[H, B] = \gamma B$	$[H, P] = P$	$[H, Z] = (\gamma + 1)Z$			$\gamma \in (-1, 1)$
121	$[B, P] = Z$	$[H, B] = B + P$	$[H, P] = P$	$[H, Z] = 2Z$			
123	$[B, P] = Z$	$[H, B] = \alpha B + P$	$[H, P] = -B + \alpha P$	$[H, Z] = 2\alpha Z$			$\alpha > 0$
135	$[B, P] = Z$	$[Z, B] = P$	$[H, P] = P$	$[H, Z] = Z$	$[B, B] = R$	$co(4) \ltimes \mathbb{R}^4$	
135	$[B, P] = Z$	$[Z, B] = -P$	$[H, P] = P$	$[H, Z] = Z$	$[B, B] = -R$	$co(3, 1) \ltimes \mathbb{R}^{3,1}$	

5. CONCLUSIONS

We have presented a deformation theory approach to the classification of kinematical Lie algebras (in $3 + 1$ dimensions) as deformations of the static kinematical Lie algebra: the one where all brackets except those which define it as a kinematical Lie algebra are zero. We saw that all deformations of the static Lie algebra are necessarily kinematical. This recovers the classical result of Bacry and Nuyts [2]. The static kinematical Lie algebra admits a one-dimensional central extension and we also determine all deformations of that algebra. In the process we recover some known Lie algebras – namely, those which are (trivial or non-trivial) central extensions of kinematical Lie algebras – but also some Lie algebras which are *non-central* extensions of kinematical Lie algebras. This should not come as a surprise, since deformation and central extension do not commute, hence there is no reason to expect that deforming the central extension of a Lie algebra \mathfrak{g} one should recover the central extension of a deformation of \mathfrak{g} .

The results are summarised in two tables: Table 1 contains the kinematical Lie algebras and is to be compared with Table 1 in [2], whereas Table 2 contains the deformations of the centrally extended static kinematical Lie algebras. The notation employed in these tables is an abbreviated notation borrowed from [2].

This paper lays the groundwork for two companion papers: one [5] where we obtain the analogous classifications as in this paper but in dimension $D + 1$ for all $D \geq 4$, and another [6] where we classify kinematical Lie algebras in dimension $2 + 1$. The three papers have been separated because they differ substantially in the technicalities, despite sharing a similar methodology.

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APPENDIX A. LIE ALGEBRA COHOMOLOGY

In this appendix we review very briefly the definition of Lie algebra cohomology as introduced by Chevalley and Eilenberg in [10].

A.1. Chevalley–Eilenberg complex. The cohomology of the Lie algebra of a Lie group G can be calculated using the Chevalley–Eilenberg complex, which is isomorphic to the subcomplex of the de Rham complex of G consisting of left-invariant differential forms. There is also a purely algebraic description which takes as starting data a Lie algebra and a representation.

Let \mathfrak{g} be a (finite-dimensional, real) Lie algebra and \mathfrak{m} a module. If $X \in \mathfrak{g}$ and $v \in \mathfrak{m}$, we will let $Xv \in \mathfrak{m}$ denote the action of X on v . Being a module, it satisfies $X(Yv) - Y(Xv) = [X, Y]v$, for all $X, Y \in \mathfrak{g}$ and $v \in \mathfrak{m}$. The cochains in the Chevalley–Eilenberg complex are skewsymmetric multilinear maps $\wedge^p \mathfrak{g} \rightarrow \mathfrak{m}$ where p runs from 0 to $\dim \mathfrak{g}$. Let $C^p(\mathfrak{g}; \mathfrak{m}) = \wedge^p \mathfrak{g}^* \otimes \mathfrak{m}$ denote the space of p -cochains. The differential $\partial : C^p(\mathfrak{g}; \mathfrak{m}) \rightarrow C^{p+1}(\mathfrak{g}; \mathfrak{m})$ is determined by its action on \mathfrak{g}^* and \mathfrak{m} and extending it as an odd derivation over the wedge product. If $v \in \mathfrak{m}$, then $\partial v \in \mathfrak{g}^* \otimes \mathfrak{m}$ is given by

$$\partial v(X) = Xv \quad (142)$$

and if $\alpha \in \mathfrak{g}^*$, $\partial \alpha \in \wedge^2 \mathfrak{g}^*$ is given by

$$\partial \alpha(X, Y) = -\alpha([X, Y]), \quad (143)$$

for all $X, Y \in \mathfrak{g}$. Since ∂ is an odd derivation, $\partial^2 = \frac{1}{2}[\partial, \partial]$ is an even derivation, so it is also determined by its action on generators. On $v \in \mathfrak{m}$, $\partial^2 v = 0$ using that \mathfrak{m} is a \mathfrak{g} -module, whereas on $\alpha \in \mathfrak{g}^*$, $\partial^2 \alpha = 0$ by virtue of the Jacobi identity of \mathfrak{g} . Therefore $\partial^2 = 0$.

Let X_i be a basis for \mathfrak{g} and α^i the canonically dual basis for \mathfrak{g}^* . Let $[X_i, X_j] = f_{ij}^k X_k$ define the structure constants of \mathfrak{g} relative to this choice of basis. Then we can write the differentials above as follows:

$$\partial v = \alpha^i \otimes X_i v \quad \text{and} \quad \partial \alpha^k = -\frac{1}{2} f_{ij}^k \alpha^i \wedge \alpha^j \quad (144)$$

and we extend it by

$$\partial(\alpha \wedge \beta \otimes v) = \partial \alpha \wedge \beta \otimes v + (-1)^{|\alpha|} \alpha \wedge \partial \beta \otimes v + (-1)^{|\alpha|+|\beta|} \alpha \wedge \beta \otimes \partial v, \quad (145)$$

for all homogeneous $\alpha, \beta \in \wedge^\bullet \mathfrak{g}^*$ and $v \in \mathfrak{m}$.

The relevant complex when computing Lie algebra deformations of \mathfrak{g} is $C^\bullet(\mathfrak{g}; \mathfrak{g})$ where $\mathfrak{m} = \mathfrak{g}$ is the adjoint representation. In this case the first three differentials $\partial : C^p(\mathfrak{g}; \mathfrak{g}) \rightarrow C^{p+1}(\mathfrak{g}; \mathfrak{g})$ for $p = 0, 1, 2$ are given explicitly, for $X, Y, Z \in \mathfrak{g}$, $\beta \in C^1(\mathfrak{g}; \mathfrak{g})$ and $\mu \in C^2(\mathfrak{g}; \mathfrak{g})$, by

$$\begin{aligned} \partial X(Y) &= -[X, Y] \\ \partial \beta(X, Y) &= [X, \beta(Y)] - [Y, \beta(X)] - \beta([X, Y]) \\ \partial \mu(X, Y, Z) &= [X, \mu(Y, Z)] - \mu([X, Y], Z) + \text{cyclic}. \end{aligned} \quad (146)$$

In this paper, however, we are interested not in all Lie algebra deformations, but only in deformations within the class of kinematical Lie algebras. The complex $C^\bullet(\mathfrak{g}; \mathfrak{g})$ seems too big at face value and we should work instead with a **relative** subcomplex. Let \mathfrak{s} be a Lie subalgebra of \mathfrak{g} (in the case which interests us in this paper, $\mathfrak{s} \cong \mathfrak{so}(3)$ is the rotational subalgebra). We limit ourselves to deformations where the brackets involving \mathfrak{s} are not modified. This means that if $\varphi \in C^2(\mathfrak{g}; \mathfrak{g})$ is the deformation, we require that $\iota_X \varphi = 0$ for all $X \in \mathfrak{s}$ and we also require that φ be \mathfrak{s} -invariant, which follows from the Jacobi identity involving one element from \mathfrak{s} . These two conditions are equivalent to $\iota_X \alpha = 0$ and $\iota_X \partial \alpha = 0$ for all $X \in \mathfrak{s}$, which defines the relative subcomplex $C^\bullet(\mathfrak{g}, \mathfrak{s}; \mathfrak{g})$. For the static kinematical Lie algebra \mathfrak{g} (and also for its universal central extension), the rotational subalgebra \mathfrak{s} has a complementary ideal \mathfrak{h} and then the relative subcomplex $C^\bullet(\mathfrak{g}, \mathfrak{s}; \mathfrak{g})$ is isomorphic to the subcomplex $C^\bullet(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ consisting of the \mathfrak{s} -invariant elements of the Chevalley–Eilenberg complex of the Lie algebra \mathfrak{h} relative to the representation \mathfrak{g} . As we will now briefly recall, a celebrated theorem of Hochschild and Serre says that there is a close relation between the cohomology of $C^\bullet(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$ and of $C^\bullet(\mathfrak{g}; \mathfrak{g})$. In particular, for the static kinematical Lie algebra (and also for its universal central extension), every deformation will necessarily be kinematical.

A.2. The Hochschild–Serre spectral sequence. In [11] Hochschild and Serre proved a factorisation theorem that in many cases simplifies the calculation of Lie algebra cohomology groups. Let \mathfrak{g} be a finite-dimensional real Lie algebra and \mathfrak{h} an ideal such that the quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{h}$ is semisimple. Let \mathfrak{m} denote a \mathfrak{g} -module, which is then also an \mathfrak{h} -module. Hochschild and Serre use the ideal \mathfrak{h} to define

a filtration of the cochains $C^\bullet(\mathfrak{g}; \mathfrak{m})$, whose associated spectral sequence degenerates at the second page yielding the following isomorphism:

$$H^n(\mathfrak{g}; \mathfrak{m}) \cong \bigoplus_{i=0}^n H^{n-i}(\mathfrak{s}; \mathbb{R}) \otimes H^i(\mathfrak{h}; \mathfrak{m})^{\mathfrak{s}}, \quad (147)$$

where the superscript \mathfrak{s} denotes \mathfrak{s} -invariants. Since \mathfrak{s} is semisimple, it acts reducibly on the cochains $C^\bullet(\mathfrak{h}; \mathfrak{m})$ and hence the \mathfrak{s} -invariant cohomology can be computed from the \mathfrak{s} -invariant cochains.

Moreover, from the Whitehead lemmas (see, e.g., [12, §III.10]), $H^1(\mathfrak{s}; \mathbb{R}) = H^2(\mathfrak{s}; \mathbb{R}) = 0$. If, in addition, \mathfrak{s} is simple then $H^3(\mathfrak{s}; \mathbb{R}) \cong \mathbb{R}$. Hence for \mathfrak{s} simple, the first few $H^\bullet(\mathfrak{g}; \mathfrak{g})$ are as follows

$$\begin{aligned} H^0(\mathfrak{g}; \mathfrak{g}) &\cong Z(\mathfrak{g}) & H^2(\mathfrak{g}; \mathfrak{g}) &\cong H^2(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}} \\ H^1(\mathfrak{g}; \mathfrak{g}) &\cong H^1(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}} & H^3(\mathfrak{g}; \mathfrak{g}) &\cong H^3(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}} \oplus Z(\mathfrak{g}), \end{aligned} \quad (148)$$

where $Z(\mathfrak{g})$ denotes the center of \mathfrak{g} . In particular, the infinitesimal deformations of \mathfrak{g} are such that the brackets involving \mathfrak{s} are not modified. This, of course, is a consequence of the well-known rigidity of semisimple Lie algebras and their finite-dimensional modules.

A.3. Central extension of the static kinematical Lie algebra. As an application of the Hochschild–Serre factorisation theorem, let us calculate the universal central extension of the static kinematical Lie algebra.

The static kinematical Lie algebra \mathfrak{g} is a ten-dimensional Lie algebra with generators R_i, B_i, P_i and H , with the following nonzero Lie brackets:

$$[R_i, R_j] = \epsilon_{ijk} R_k \quad [R_i, B_j] = \epsilon_{ijk} B_k \quad \text{and} \quad [R_i, P_j] = \epsilon_{ijk} P_k. \quad (149)$$

In other words, \mathfrak{g} is isomorphic to the semi-direct product of the simple Lie algebra $\mathfrak{so}(3)$ (spanned by the R_i) and an abelian Lie algebra transforming as the representation $2V \oplus \mathbb{R}$, where V is the 3-dimensional vector representation and \mathbb{R} is the trivial representation. It is often convenient to abbreviate the Lie bracket as follows:

$$[R, R] = R \quad [R, B] = B \quad \text{and} \quad [R, P] = P, \quad (150)$$

which does not lead to any ambiguity as there is (up to scale) only one $\mathfrak{so}(3)$ -equivariant map $V \otimes V \rightarrow V$.

Central extensions of \mathfrak{g} are classified by the second Chevalley–Eilenberg cohomology group $H^2(\mathfrak{g}; \mathbb{R})$, which by Hochschild–Serre is isomorphic to $H^2(\mathfrak{h}; \mathbb{R})^{\mathfrak{s}}$, where \mathfrak{h} is the abelian ideal generated by B_i, P_i, H and $\mathfrak{s} \cong \mathfrak{so}(3)$ is the simple subalgebra generated by R_i . Since \mathfrak{s} is simple, and hence reductive, we may calculate the \mathfrak{s} -invariant cohomology from the \mathfrak{s} -invariant subcomplex: $C^p(\mathfrak{h}; \mathbb{R})^{\mathfrak{s}} = \text{Hom}_{\mathfrak{s}}(\wedge^p \mathfrak{h}, \mathbb{R})$. By inspection, $C^p(\mathfrak{h}; \mathbb{R})^{\mathfrak{s}}$, for $p = 1, 2$, are one-dimensional with basis η and $\pi^i \wedge \beta^i$, respectively, where β^i, π^i, η are the basis for \mathfrak{h}^* canonically dual to the basis B_i, P_i, H for \mathfrak{h} . Since \mathfrak{h} is abelian, the Chevalley–Eilenberg differential is identically zero and hence

$$H^2(\mathfrak{h}; \mathbb{R})^{\mathfrak{s}} \cong \mathbb{R} \langle [\pi^i \wedge \beta^i] \rangle, \quad (151)$$

so that there is a one-dimensional universal central extension with Lie bracket

$$[B_i, P_j] = \delta_{ij} Z \quad (\text{abbreviated as } [B, P] = Z) \quad (152)$$

where Z is the central generator.

APPENDIX B. COCHAINS FOR THE STATIC KINEMATICAL LIE ALGEBRA

In this appendix we list the relevant cochains in the complex calculating $H^2(\mathfrak{g}; \mathfrak{g})$ for the static kinematical Lie algebra \mathfrak{g} with basis R_i, B_i, P_i, H . The ideal \mathfrak{h} is spanned by B_i, P_i, H with simple quotient \mathfrak{s} , isomorphic to the subalgebra generated by R_i . The canonical dual basis for \mathfrak{h}^* is given by β^i, π^i, η . By Hochschild–Serre, it suffices to calculate the cohomology of the \mathfrak{s} -invariant complex $C^\bullet(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$. The relevant cochains are tabulated below using an abbreviated notation where we have omitted \otimes, \wedge and any indices. For example, $\beta R = \beta^i \otimes R_i$, $\frac{1}{2} \beta \beta R = \frac{1}{2} \epsilon_{ijk} \beta^i \wedge \beta^j \otimes R_k$ and $\beta \pi \pi B = \beta^i \wedge \pi^i \wedge \pi^j \otimes B_j$.

TABLE 3. Basis for $C^1(\mathfrak{h}; \mathfrak{g})^{\mathfrak{s}}$

α_1	α_2	α_3	α_4	α_5	α_6	α_7
βR	βB	βP	πR	πB	πP	ηH

The Chevalley–Eilenberg differential is defined on generators in such a way that it is zero except for

$$\partial R_i = -\epsilon_{ijk} (\beta^j B_k + \pi^j P_k), \quad (153)$$

TABLE 4. Basis for $C^2(\mathfrak{h}; \mathfrak{g})^s$

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$\beta\pi H$	$\eta\beta R$	$\eta\beta B$	$\eta\beta P$	$\eta\pi R$	$\eta\pi B$	$\eta\pi P$	$\frac{1}{2}\beta\beta R$
c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}
$\frac{1}{2}\beta\beta B$	$\frac{1}{2}\beta\beta P$	$\beta\pi R$	$\beta\pi B$	$\beta\pi P$	$\frac{1}{2}\pi\pi R$	$\frac{1}{2}\pi\pi B$	$\frac{1}{2}\pi\pi P$

TABLE 5. Basis for $C^3(\mathfrak{h}; \mathfrak{g})^s$

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
$\beta\pi\beta P$	$\eta\beta\beta B$	$\eta\beta\pi P$	$\eta\beta\pi B$	$\eta\pi\pi P$	$\eta\beta\beta P$	$\eta\beta\beta R$	$\beta\pi\pi P$	$\beta\pi\beta B$	$\eta\beta\pi R$
b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}
$\beta\pi\pi B$	$\beta\beta\beta H$	$\eta\beta\pi H$	$\eta\pi\pi B$	$\beta\pi\beta R$	$\beta\pi\pi R$	$\eta\pi\pi R$	$\beta\beta\pi H$	$\beta\pi\pi H$	$\pi\pi\pi H$

from where we can calculate the differential on cochains. Using the notation in the above tables of cochains, the nonzero differentials are:

$$\begin{aligned} \partial a_1 &= 2c_9 + c_{13} & \partial c_2 &= -b_2 - b_3 & \partial c_8 &= b_1 & \partial c_{14} &= -b_{11}. \\ \partial a_4 &= c_{12} + 2c_{16} & \partial c_5 &= -b_4 - b_5 & \partial c_{11} &= b_8 - b_9 \end{aligned} \quad (154)$$

Finally, we work out (the restriction of) the Nijenhuis–Richardson bracket

$$[-, -] : C^2(\mathfrak{h}; \mathfrak{g})^s \times C^2(\mathfrak{h}; \mathfrak{g})^s \rightarrow C^3(\mathfrak{h}; \mathfrak{g})^s \quad (155)$$

on the above basis of cochains:

$$\begin{aligned} \llbracket c_1, c_2 \rrbracket &= b_{15} & \llbracket c_3, c_8 \rrbracket &= b_7 & \llbracket c_6, c_8 \rrbracket &= b_{10} & \llbracket c_9, c_{11} \rrbracket &= -b_{15} \\ \llbracket c_1, c_3 \rrbracket &= b_9 + b_{13} & \llbracket c_3, c_9 \rrbracket &= \frac{3}{2}b_2 & \llbracket c_6, c_9 \rrbracket &= b_4 & \llbracket c_9, c_{13} \rrbracket &= -b_1 \\ \llbracket c_1, c_4 \rrbracket &= b_1 & \llbracket c_3, c_{10} \rrbracket &= b_6 & \llbracket c_6, c_{10} \rrbracket &= b_3 - \frac{1}{2}b_2 & \llbracket c_9, c_{15} \rrbracket &= b_{11} \\ \llbracket c_1, c_5 \rrbracket &= b_{16} & \llbracket c_3, c_{11} \rrbracket &= b_{10} & \llbracket c_6, c_{11} \rrbracket &= b_{17} & \llbracket c_{10}, c_{12} \rrbracket &= b_1 \\ \llbracket c_1, c_6 \rrbracket &= b_{11} & \llbracket c_3, c_{13} \rrbracket &= b_3 & \llbracket c_6, c_{12} \rrbracket &= b_{14} & \llbracket c_{10}, c_{14} \rrbracket &= -b_{15} \\ \llbracket c_1, c_7 \rrbracket &= b_8 + b_{13} & \llbracket c_3, c_{15} \rrbracket &= -\frac{1}{2}b_{14} & \llbracket c_6, c_{13} \rrbracket &= b_5 - b_4 & \llbracket c_{10}, c_{15} \rrbracket &= b_8 - b_9 \\ \llbracket c_1, c_9 \rrbracket &= \frac{1}{2}b_{18} & \llbracket c_4, c_9 \rrbracket &= \frac{1}{2}b_6 & \llbracket c_6, c_{16} \rrbracket &= -\frac{1}{2}b_{14} & \llbracket c_{10}, c_{16} \rrbracket &= -b_1 \\ \llbracket c_1, c_{10} \rrbracket &= -\frac{1}{2}b_{12} & \llbracket c_4, c_{11} \rrbracket &= b_7 & \llbracket c_7, c_{10} \rrbracket &= -\frac{1}{2}b_6 & \llbracket c_{11}, c_{12} \rrbracket &= -b_{16} \\ \llbracket c_1, c_{12} \rrbracket &= b_{19} & \llbracket c_4, c_{12} \rrbracket &= b_2 - b_3 & \llbracket c_7, c_{11} \rrbracket &= b_{10} & \llbracket c_{11}, c_{13} \rrbracket &= b_{15} \\ \llbracket c_1, c_{13} \rrbracket &= b_{18} & \llbracket c_4, c_{13} \rrbracket &= b_6 & \llbracket c_7, c_{12} \rrbracket &= b_4 & \llbracket c_{11}, c_{16} \rrbracket &= b_{16} \\ \llbracket c_1, c_{15} \rrbracket &= b_{20} & \llbracket c_4, c_{14} \rrbracket &= b_{10} & \llbracket c_7, c_{14} \rrbracket &= b_{17} & \llbracket c_{12}, c_{12} \rrbracket &= -2b_{11} \\ \llbracket c_1, c_{16} \rrbracket &= -\frac{1}{2}b_{20} & \llbracket c_4, c_{15} \rrbracket &= b_4 - \frac{1}{2}b_5 & \llbracket c_7, c_{15} \rrbracket &= b_{14} & \llbracket c_{12}, c_{13} \rrbracket &= -b_8 + b_9 \\ \llbracket c_2, c_9 \rrbracket &= \frac{1}{2}b_7 & \llbracket c_4, c_{16} \rrbracket &= b_3 & \llbracket c_7, c_{16} \rrbracket &= \frac{1}{2}b_5 & \llbracket c_{12}, c_{16} \rrbracket &= b_{11} \\ \llbracket c_2, c_{12} \rrbracket &= -b_{10} & \llbracket c_5, c_{10} \rrbracket &= -\frac{1}{2}b_7 & \llbracket c_8, c_{12} \rrbracket &= b_{15} & \llbracket c_{13}, c_{13} \rrbracket &= 2b_1 \\ \llbracket c_2, c_{15} \rrbracket &= -\frac{1}{2}b_{17} & \llbracket c_5, c_{13} \rrbracket &= -b_{10} & \llbracket c_8, c_{15} \rrbracket &= b_{16} & \llbracket c_{13}, c_{14} \rrbracket &= -b_{16} \\ & & \llbracket c_5, c_{16} \rrbracket &= -\frac{1}{2}b_{17} & & & \llbracket c_{13}, c_{15} \rrbracket &= -b_{11} \end{aligned} \quad (156)$$

APPENDIX C. COCHAINS FOR THE CENTRALLY EXTENDED STATIC KINEMATICAL LIE ALGEBRA

In this appendix we list the relevant cochains in the complex calculating $H^2(\tilde{\mathfrak{g}}; \tilde{\mathfrak{g}})$ for the centrally extended static kinematical Lie algebra $\tilde{\mathfrak{g}}$ with basis R_i, B_i, P_i, H, Z . The ideal $\tilde{\mathfrak{h}}$ is spanned by B_i, P_i, H, Z with simple quotient \mathfrak{s} , isomorphic to the subalgebra generated by R_i . The canonical dual basis for $\tilde{\mathfrak{h}}^*$ is given by $\beta^i, \pi^i, \eta, \zeta$. By Hochschild–Serre, it suffices to calculate the cohomology of the \mathfrak{s} -invariant complex $C^\bullet(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^s$. The relevant cochains are tabulated below using the same abbreviated notation as in the previous appendix.

TABLE 6. Basis for $C^1(\tilde{\mathfrak{h}}; \tilde{\mathfrak{g}})^s$

\tilde{a}_1	\tilde{a}_2	\tilde{a}_3	\tilde{a}_4	\tilde{a}_5	\tilde{a}_6	\tilde{a}_7	\tilde{a}_8	\tilde{a}_9	\tilde{a}_{10}
ζH	ζZ	ηH	ηZ	βR	βB	βP	πR	πB	πP

TABLE 7. Basis for $C^2(\mathfrak{h}; \tilde{\mathfrak{g}})^s$

\tilde{c}_1	\tilde{c}_2	\tilde{c}_3	\tilde{c}_4	\tilde{c}_5	\tilde{c}_6	\tilde{c}_7	\tilde{c}_8	\tilde{c}_9	\tilde{c}_{10}	\tilde{c}_{11}	\tilde{c}_{12}	\tilde{c}_{13}
$\beta\pi Z$	$\beta\pi H$	$\beta\pi R$	$\beta\pi B$	$\beta\pi P$	$\frac{1}{2}\beta\beta R$	$\frac{1}{2}\beta\beta B$	$\frac{1}{2}\beta\beta P$	$\frac{1}{2}\pi\pi R$	$\frac{1}{2}\pi\pi B$	$\frac{1}{2}\pi\pi P$	$\eta\pi R$	$\eta\pi B$
\tilde{c}_{14}	\tilde{c}_{15}	\tilde{c}_{16}	\tilde{c}_{17}	\tilde{c}_{18}	\tilde{c}_{19}	\tilde{c}_{20}	\tilde{c}_{21}	\tilde{c}_{22}	\tilde{c}_{23}	\tilde{c}_{24}	\tilde{c}_{25}	
$\eta\pi P$	$\eta\beta R$	$\eta\beta B$	$\eta\beta P$	$\zeta\pi R$	$\zeta\pi B$	$\zeta\pi P$	$\zeta\beta R$	$\zeta\beta B$	$\zeta\beta P$	$\eta\zeta Z$	$\eta\zeta H$	

TABLE 8. Basis for $C^3(\mathfrak{h}; \tilde{\mathfrak{g}})^s$

\tilde{b}_1	\tilde{b}_2	\tilde{b}_3	\tilde{b}_4	\tilde{b}_5	\tilde{b}_6	\tilde{b}_7	\tilde{b}_8	\tilde{b}_9	\tilde{b}_{10}	\tilde{b}_{11}
$\eta\zeta\beta R$	$\eta\zeta\beta B$	$\eta\zeta\beta P$	$\eta\zeta\pi R$	$\eta\zeta\pi B$	$\eta\zeta\pi P$	$\eta\beta\pi H$	$\eta\beta\pi Z$	$\zeta\beta\pi H$	$\zeta\beta\pi Z$	$\beta\beta\beta H$
\tilde{b}_{12}	\tilde{b}_{13}	\tilde{b}_{14}	\tilde{b}_{15}	\tilde{b}_{16}	\tilde{b}_{17}	\tilde{b}_{18}	\tilde{b}_{19}	\tilde{b}_{20}	\tilde{b}_{21}	\tilde{b}_{22}
$\beta\beta\beta Z$	$\beta\beta\pi H$	$\beta\beta\pi Z$	$\beta\pi\pi H$	$\beta\pi\pi Z$	$\pi\pi\pi H$	$\pi\pi\pi Z$	$\eta\beta\beta R$	$\eta\beta\beta B$	$\eta\beta\beta P$	$\eta\beta\pi R$
\tilde{b}_{23}	\tilde{b}_{24}	\tilde{b}_{25}	\tilde{b}_{26}	\tilde{b}_{27}	\tilde{b}_{28}	\tilde{b}_{29}	\tilde{b}_{30}	\tilde{b}_{31}	\tilde{b}_{32}	\tilde{b}_{33}
$\eta\beta\pi B$	$\eta\beta\pi P$	$\eta\pi\pi R$	$\eta\pi\pi B$	$\eta\pi\pi P$	$\zeta\beta\beta R$	$\zeta\beta\beta B$	$\zeta\beta\beta P$	$\zeta\beta\pi R$	$\zeta\beta\pi B$	$\zeta\beta\pi P$
\tilde{b}_{34}	\tilde{b}_{35}	\tilde{b}_{36}	\tilde{b}_{37}	\tilde{b}_{38}	\tilde{b}_{39}	\tilde{b}_{40}	\tilde{b}_{41}	\tilde{b}_{42}		
$\zeta\pi\pi R$	$\zeta\pi\pi B$	$\zeta\pi\pi P$	$\beta\pi\beta R$	$\beta\pi\beta B$	$\beta\pi\beta P$	$\beta\pi\pi R$	$\beta\pi\pi B$	$\beta\pi\pi P$		

The Chevalley–Eilenberg differential is defined on generators by

$$\partial\zeta = -\beta^i\pi^i \quad \partial B_i = -\pi^i Z \quad \partial P_i = \beta^i Z \quad \text{and} \quad \partial R_i = -\epsilon_{ijk}(\beta^j B_k + \pi^j P_k), \quad (157)$$

and zero on anything else. From these we can calculate the differential on cochains. Using the notation in the above tables of cochains, the nonzero differentials are:

$$\begin{aligned}
\partial\tilde{a}_1 &= -\tilde{c}_2 & \partial\tilde{c}_3 &= -\tilde{b}_{38} - \tilde{b}_{42} & \partial\tilde{c}_{10} &= -\frac{1}{2}\tilde{b}_{18} & \partial\tilde{c}_{19} &= -\tilde{b}_{41} \\
\partial\tilde{a}_2 &= -\tilde{c}_1 & \partial\tilde{c}_4 &= -\tilde{b}_{16} & \partial\tilde{c}_{11} &= \frac{1}{2}\tilde{b}_{16} & \partial\tilde{c}_{20} &= -\tilde{b}_{10} - \tilde{b}_{42} \\
\partial\tilde{a}_5 &= \tilde{c}_5 + 2\tilde{c}_7 & \partial\tilde{c}_5 &= \tilde{b}_{14} & \partial\tilde{c}_{12} &= -\tilde{b}_{23} - \tilde{b}_{27} & \partial\tilde{c}_{21} &= -\tilde{b}_{29} - \tilde{b}_{33} - \tilde{b}_{37} \\
\partial\tilde{a}_6 &= \tilde{c}_1 & \partial\tilde{c}_6 &= \tilde{b}_{39} & \partial\tilde{c}_{14} &= -\tilde{b}_8 & \partial\tilde{c}_{22} &= -\tilde{b}_{10} - \tilde{b}_{38} \\
\partial\tilde{a}_8 &= \tilde{c}_4 + 2\tilde{c}_{11} & \partial\tilde{c}_7 &= -\frac{1}{2}\tilde{b}_{14} & \partial\tilde{c}_{15} &= -\tilde{b}_{20} - \tilde{b}_{24} & \partial\tilde{c}_{23} &= -\tilde{b}_{39} \\
\partial\tilde{a}_{10} &= \tilde{c}_1 & \partial\tilde{c}_8 &= \frac{1}{2}\tilde{b}_{12} & \partial\tilde{c}_{16} &= -\tilde{b}_8 & \partial\tilde{c}_{24} &= \tilde{b}_8 \\
& & \partial\tilde{c}_9 &= -\tilde{b}_{41} & \partial\tilde{c}_{18} &= -\tilde{b}_{32} - \tilde{b}_{36} - \tilde{b}_{40} & \partial\tilde{c}_{25} &= \tilde{b}_7.
\end{aligned} \quad (158)$$

Finally, we work out (the restriction of) the Nijenhuis–Richardson bracket

$$[-, -] : C^2(\mathfrak{h}; \mathfrak{g})^s \times C^2(\mathfrak{h}; \mathfrak{g})^s \rightarrow C^3(\mathfrak{h}; \mathfrak{g})^s \quad (159)$$

on the above basis of cochains. Although not all of the brackets appear in our calculations, we list the nonzero ones here for completeness and in order to allow others to reproduce our calculations.

$$\begin{aligned}
\llbracket \tilde{c}_1, \tilde{c}_4 \rrbracket &= \tilde{b}_{16} & \llbracket \tilde{c}_2, \tilde{c}_{24} \rrbracket &= \tilde{b}_{10} & \llbracket \tilde{c}_5, \tilde{c}_{10} \rrbracket &= -\tilde{b}_{41} & \llbracket \tilde{c}_{12}, \tilde{c}_{23} \rrbracket &= -\tilde{b}_1 \\
\llbracket \tilde{c}_1, \tilde{c}_5 \rrbracket &= -\tilde{b}_{14} & \llbracket \tilde{c}_2, \tilde{c}_{25} \rrbracket &= \tilde{b}_9 & \llbracket \tilde{c}_5, \tilde{c}_{11} \rrbracket &= \tilde{b}_{39} - \tilde{b}_{42} & \llbracket \tilde{c}_8, \tilde{c}_{14} \rrbracket &= -\frac{1}{2}\tilde{b}_{21} & \llbracket \tilde{c}_{12}, \tilde{c}_{25} \rrbracket &= \tilde{b}_4 \\
\llbracket \tilde{c}_1, \tilde{c}_7 \rrbracket &= \frac{1}{2}\tilde{b}_{14} & \llbracket \tilde{c}_3, \tilde{c}_4 \rrbracket &= -\tilde{b}_{40} & \llbracket \tilde{c}_5, \tilde{c}_{12} \rrbracket &= -\tilde{b}_{22} & \llbracket \tilde{c}_8, \tilde{c}_{16} \rrbracket &= \tilde{b}_{21} & \llbracket \tilde{c}_{13}, \tilde{c}_{20} \rrbracket &= -\tilde{b}_5 \\
\llbracket \tilde{c}_1, \tilde{c}_8 \rrbracket &= -\frac{1}{2}\tilde{b}_{12} & \llbracket \tilde{c}_3, \tilde{c}_5 \rrbracket &= \tilde{b}_{37} & \llbracket \tilde{c}_5, \tilde{c}_{13} \rrbracket &= \tilde{b}_{27} - \tilde{b}_{23} & \llbracket \tilde{c}_8, \tilde{c}_{18} \rrbracket &= -\frac{1}{2}\tilde{b}_{28} & \llbracket \tilde{c}_{13}, \tilde{c}_{21} \rrbracket &= \tilde{b}_4 \\
\llbracket \tilde{c}_1, \tilde{c}_{10} \rrbracket &= \frac{1}{2}\tilde{b}_{18} & \llbracket \tilde{c}_3, \tilde{c}_7 \rrbracket &= -\tilde{b}_{37} & \llbracket \tilde{c}_5, \tilde{c}_{16} \rrbracket &= \tilde{b}_{24} & \llbracket \tilde{c}_8, \tilde{c}_{19} \rrbracket &= \tilde{b}_{33} - \frac{1}{2}\tilde{b}_{29} & \llbracket \tilde{c}_{13}, \tilde{c}_{22} \rrbracket &= \tilde{b}_5 \\
\llbracket \tilde{c}_1, \tilde{c}_{11} \rrbracket &= -\frac{1}{2}\tilde{b}_{16} & \llbracket \tilde{c}_3, \tilde{c}_{11} \rrbracket &= \tilde{b}_{37} & \llbracket \tilde{c}_5, \tilde{c}_{17} \rrbracket &= \tilde{b}_{21} & \llbracket \tilde{c}_8, \tilde{c}_{20} \rrbracket &= -\frac{1}{2}\tilde{b}_{30} & \llbracket \tilde{c}_{13}, \tilde{c}_{23} \rrbracket &= \tilde{b}_6 - \tilde{b}_2 \\
\llbracket \tilde{c}_1, \tilde{c}_{14} \rrbracket &= \tilde{b}_8 & \llbracket \tilde{c}_3, \tilde{c}_{13} \rrbracket &= \tilde{b}_{25} & \llbracket \tilde{c}_5, \tilde{c}_{18} \rrbracket &= -\tilde{b}_{31} & \llbracket \tilde{c}_8, \tilde{c}_{22} \rrbracket &= \tilde{b}_{30} & \llbracket \tilde{c}_{13}, \tilde{c}_{25} \rrbracket &= \tilde{b}_5 \\
\llbracket \tilde{c}_1, \tilde{c}_{16} \rrbracket &= \tilde{b}_8 & \llbracket \tilde{c}_3, \tilde{c}_{14} \rrbracket &= \tilde{b}_{22} & \llbracket \tilde{c}_5, \tilde{c}_{19} \rrbracket &= \tilde{b}_{36} - \tilde{b}_{32} & \llbracket \tilde{c}_9, \tilde{c}_{14} \rrbracket &= \tilde{b}_{25} & \llbracket \tilde{c}_{14}, \tilde{c}_{18} \rrbracket &= \tilde{b}_4 \\
\llbracket \tilde{c}_1, \tilde{c}_{18} \rrbracket &= \tilde{b}_{40} & \llbracket \tilde{c}_3, \tilde{c}_{16} \rrbracket &= \tilde{b}_{22} & \llbracket \tilde{c}_5, \tilde{c}_{22} \rrbracket &= \tilde{b}_{33} & \llbracket \tilde{c}_9, \tilde{c}_{17} \rrbracket &= \tilde{b}_{22} & \llbracket \tilde{c}_{14}, \tilde{c}_{19} \rrbracket &= \tilde{b}_5 \\
\llbracket \tilde{c}_1, \tilde{c}_{19} \rrbracket &= \tilde{b}_{41} & \llbracket \tilde{c}_3, \tilde{c}_{17} \rrbracket &= \tilde{b}_{19} & \llbracket \tilde{c}_5, \tilde{c}_{23} \rrbracket &= \tilde{b}_{30} & \llbracket \tilde{c}_9, \tilde{c}_{20} \rrbracket &= \tilde{b}_{34} & \llbracket \tilde{c}_{14}, \tilde{c}_{23} \rrbracket &= -\tilde{b}_3 \\
\llbracket \tilde{c}_1, \tilde{c}_{20} \rrbracket &= \tilde{b}_{10} + \tilde{b}_{42} & \llbracket \tilde{c}_3, \tilde{c}_{19} \rrbracket &= \tilde{b}_{34} & \llbracket \tilde{c}_6, \tilde{c}_{10} \rrbracket &= \tilde{b}_{40} & \llbracket \tilde{c}_9, \tilde{c}_{23} \rrbracket &= \tilde{b}_{31} & \llbracket \tilde{c}_{14}, \tilde{c}_{25} \rrbracket &= \tilde{b}_6 \\
\llbracket \tilde{c}_1, \tilde{c}_{21} \rrbracket &= \tilde{b}_{37} & \llbracket \tilde{c}_3, \tilde{c}_{20} \rrbracket &= \tilde{b}_{31} & \llbracket \tilde{c}_6, \tilde{c}_{13} \rrbracket &= \tilde{b}_{22} & \llbracket \tilde{c}_{10}, \tilde{c}_{14} \rrbracket &= \tilde{b}_{26} & \llbracket \tilde{c}_{15}, \tilde{c}_{19} \rrbracket &= -\tilde{b}_4 \\
\llbracket \tilde{c}_1, \tilde{c}_{22} \rrbracket &= \tilde{b}_{10} + \tilde{b}_{38} & \llbracket \tilde{c}_3, \tilde{c}_{22} \rrbracket &= \tilde{b}_{31} & \llbracket \tilde{c}_6, \tilde{c}_{16} \rrbracket &= \tilde{b}_{19} & \llbracket \tilde{c}_{10}, \tilde{c}_{15} \rrbracket &= -\frac{1}{2}\tilde{b}_{25} & \llbracket \tilde{c}_{15}, \tilde{c}_{22} \rrbracket &= -\tilde{b}_1 \\
\llbracket \tilde{c}_1, \tilde{c}_{23} \rrbracket &= \tilde{b}_{39} & \llbracket \tilde{c}_3, \tilde{c}_{23} \rrbracket &= \tilde{b}_{28} & \llbracket \tilde{c}_6, \tilde{c}_{19} \rrbracket &= \tilde{b}_{31} & \llbracket \tilde{c}_{10}, \tilde{c}_{16} \rrbracket &= -\frac{1}{2}\tilde{b}_{26} & \llbracket \tilde{c}_{15}, \tilde{c}_{25} \rrbracket &= \tilde{b}_1 \\
\llbracket \tilde{c}_1, \tilde{c}_{24} \rrbracket &= -\tilde{b}_8 & \llbracket \tilde{c}_4, \tilde{c}_4 \rrbracket &= -2\tilde{b}_{41} & \llbracket \tilde{c}_6, \tilde{c}_{22} \rrbracket &= \tilde{b}_{28} & \llbracket \tilde{c}_{10}, \tilde{c}_{17} \rrbracket &= \tilde{b}_{23} - \frac{1}{2}\tilde{b}_{27} & \llbracket \tilde{c}_{16}, \tilde{c}_{19} \rrbracket &= -\tilde{b}_5 \\
\llbracket \tilde{c}_1, \tilde{c}_{25} \rrbracket &= -\tilde{b}_7 & \llbracket \tilde{c}_4, \tilde{c}_5 \rrbracket &= \tilde{b}_{38} - \tilde{b}_{42} & \llbracket \tilde{c}_7, \tilde{c}_{10} \rrbracket &= \tilde{b}_{41} & \llbracket \tilde{c}_{10}, \tilde{c}_{20} \rrbracket &= \tilde{b}_{35} & \llbracket \tilde{c}_{16}, \tilde{c}_{21} \rrbracket &= \tilde{b}_1 \\
\llbracket \tilde{c}_2, \tilde{c}_4 \rrbracket &= \tilde{b}_{15} & \llbracket \tilde{c}_4, \tilde{c}_6 \rrbracket &= \tilde{b}_{37} & \llbracket \tilde{c}_7, \tilde{c}_{13} \rrbracket &= \tilde{b}_{23} & \llbracket \tilde{c}_{10}, \tilde{c}_{21} \rrbracket &= -\frac{1}{2}\tilde{b}_{34} & \llbracket \tilde{c}_{16}, \tilde{c}_{23} \rrbracket &= \tilde{b}_3 \\
\llbracket \tilde{c}_2, \tilde{c}_5 \rrbracket &= -\tilde{b}_{13} & \llbracket \tilde{c}_4, \tilde{c}_8 \rrbracket &= \tilde{b}_{39} & \llbracket \tilde{c}_7, \tilde{c}_{15} \rrbracket &= \frac{1}{2}\tilde{b}_{19} & \llbracket \tilde{c}_{10}, \tilde{c}_{22} \rrbracket &= -\frac{1}{2}\tilde{b}_{35} & \llbracket \tilde{c}_{16}, \tilde{c}_{25} \rrbracket &= \tilde{b}_2 \\
\llbracket \tilde{c}_2, \tilde{c}_7 \rrbracket &= \frac{1}{2}\tilde{b}_{15} & \llbracket \tilde{c}_4, \tilde{c}_{11} \rrbracket &= \tilde{b}_{38} & \llbracket \tilde{c}_7, \tilde{c}_{16} \rrbracket &= -\frac{1}{2}\tilde{b}_{20} & \llbracket \tilde{c}_{10}, \tilde{c}_{23} \rrbracket &= \tilde{b}_{32} - \frac{1}{2}\tilde{b}_{36} & \llbracket \tilde{c}_{17}, \tilde{c}_{18} \rrbracket &= \tilde{b}_1 \\
\llbracket \tilde{c}_2, \tilde{c}_8 \rrbracket &= -\frac{1}{2}\tilde{b}_{11} & \llbracket \tilde{c}_4, \tilde{c}_{13} \rrbracket &= \tilde{b}_{26} & \llbracket \tilde{c}_7, \tilde{c}_{17} \rrbracket &= -\frac{1}{2}\tilde{b}_{21} & \llbracket \tilde{c}_{11}, \tilde{c}_{12} \rrbracket &= -\frac{1}{2}\tilde{b}_{25} & \llbracket \tilde{c}_{17}, \tilde{c}_{19} \rrbracket &= \tilde{b}_2 - \tilde{b}_6 \\
\llbracket \tilde{c}_2, \tilde{c}_{10} \rrbracket &= \frac{1}{2}\tilde{b}_{17} & \llbracket \tilde{c}_4, \tilde{c}_{14} \rrbracket &= \tilde{b}_{23} & \llbracket \tilde{c}_7, \tilde{c}_{19} \rrbracket &= \tilde{b}_{32} & \llbracket \tilde{c}_{11}, \tilde{c}_{13} \rrbracket &= -\frac{1}{2}\tilde{b}_{26} & \llbracket \tilde{c}_{17}, \tilde{c}_{20} \rrbracket &= \tilde{b}_3 \\
\llbracket \tilde{c}_2, \tilde{c}_{11} \rrbracket &= -\frac{1}{2}\tilde{b}_{15} & \llbracket \tilde{c}_4, \tilde{c}_{15} \rrbracket &= -\tilde{b}_{22} & \llbracket \tilde{c}_7, \tilde{c}_{21} \rrbracket &= -\frac{1}{2}\tilde{b}_{28} & \llbracket \tilde{c}_{11}, \tilde{c}_{14} \rrbracket &= \frac{1}{2}\tilde{b}_{27} & \llbracket \tilde{c}_{17}, \tilde{c}_{22} \rrbracket &= -\tilde{b}_3 \\
\llbracket \tilde{c}_2, \tilde{c}_{12} \rrbracket &= \tilde{b}_{40} & \llbracket \tilde{c}_4, \tilde{c}_{17} \rrbracket &= \tilde{b}_{20} - \tilde{b}_{24} & \llbracket \tilde{c}_7, \tilde{c}_{22} \rrbracket &= \frac{1}{2}\tilde{b}_{29} & \llbracket \tilde{c}_{11}, \tilde{c}_{17} \rrbracket &= \tilde{b}_{24} & \llbracket \tilde{c}_{17}, \tilde{c}_{25} \rrbracket &= \tilde{b}_3 \\
\llbracket \tilde{c}_2, \tilde{c}_{13} \rrbracket &= \tilde{b}_{41} & \llbracket \tilde{c}_4, \tilde{c}_{19} \rrbracket &= \tilde{b}_{35} & \llbracket \tilde{c}_7, \tilde{c}_{23} \rrbracket &= -\frac{1}{2}\tilde{b}_{30} & \llbracket \tilde{c}_{11}, \tilde{c}_{18} \rrbracket &= -\frac{1}{2}\tilde{b}_{34} & \llbracket \tilde{c}_{18}, \tilde{c}_{24} \rrbracket &= \tilde{b}_4 \\
\llbracket \tilde{c}_2, \tilde{c}_{14} \rrbracket &= \tilde{b}_7 + \tilde{b}_{42} & \llbracket \tilde{c}_4, \tilde{c}_{20} \rrbracket &= \tilde{b}_{32} & \llbracket \tilde{c}_8, \tilde{c}_9 \rrbracket &= -\tilde{b}_{37} & \llbracket \tilde{c}_{11}, \tilde{c}_{19} \rrbracket &= -\frac{1}{2}\tilde{b}_{35} & \llbracket \tilde{c}_{19}, \tilde{c}_{24} \rrbracket &= \tilde{b}_5 \\
\llbracket \tilde{c}_2, \tilde{c}_{15} \rrbracket &= \tilde{b}_{37} & \llbracket \tilde{c}_4, \tilde{c}_{21} \rrbracket &= -\tilde{b}_{31} & \llbracket \tilde{c}_8, \tilde{c}_{10} \rrbracket &= \tilde{b}_{42} - \tilde{b}_{38} & \llbracket \tilde{c}_{11}, \tilde{c}_{20} \rrbracket &= \frac{1}{2}\tilde{b}_{36} & \llbracket \tilde{c}_{20}, \tilde{c}_{24} \rrbracket &= \tilde{b}_6 \\
\llbracket \tilde{c}_2, \tilde{c}_{16} \rrbracket &= \tilde{b}_7 + \tilde{b}_{38} & \llbracket \tilde{c}_4, \tilde{c}_{23} \rrbracket &= \tilde{b}_{29} - \tilde{b}_{33} & \llbracket \tilde{c}_8, \tilde{c}_{11} \rrbracket &= -\tilde{b}_{39} & \llbracket \tilde{c}_{11}, \tilde{c}_{23} \rrbracket &= \tilde{b}_{33} & \llbracket \tilde{c}_{21}, \tilde{c}_{24} \rrbracket &= \tilde{b}_1 \\
\llbracket \tilde{c}_2, \tilde{c}_{17} \rrbracket &= \tilde{b}_{39} & \llbracket \tilde{c}_5, \tilde{c}_5 \rrbracket &= 2\tilde{b}_{39} & \llbracket \tilde{c}_8, \tilde{c}_{12} \rrbracket &= -\frac{1}{2}\tilde{b}_{19} & \llbracket \tilde{c}_{12}, \tilde{c}_{20} \rrbracket &= -\tilde{b}_4 & \llbracket \tilde{c}_{22}, \tilde{c}_{24} \rrbracket &= \tilde{b}_2 \\
\llbracket \tilde{c}_2, \tilde{c}_{20} \rrbracket &= \tilde{b}_9 & \llbracket \tilde{c}_5, \tilde{c}_7 \rrbracket &= -\tilde{b}_{39} & \llbracket \tilde{c}_8, \tilde{c}_{13} \rrbracket &= \tilde{b}_{24} - \frac{1}{2}\tilde{b}_{20} & & & \llbracket \tilde{c}_{23}, \tilde{c}_{24} \rrbracket &= \tilde{b}_3 \\
\llbracket \tilde{c}_2, \tilde{c}_{22} \rrbracket &= \tilde{b}_9 & \llbracket \tilde{c}_5, \tilde{c}_9 \rrbracket &= -\tilde{b}_{40} & & & & & &
\end{aligned} \tag{160}$$

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